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A FRAMEWORK FOR THE INTEGRATION OF AUDITING EVIDENCE.

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A Framework for the Integration of Auditing Evidence

by

Richard Arnold Grimlund

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We have carefully read the dissertation entitled A Framework for the Integration of Auditing Evidence submitted by Richard Arnold Grimlund in partial fulfillment of the requirements of the degree of Doctor of Philosophy and recommend its acceptance. In support of this recommendation we present the following joint statement of evaluation to be filed with the dissertation.

In recent years there has been a growing interest among auditors in the use of statistical methods for studying the reliability of a firm's financial statements. Numerous methods are available to aid the auditor in examining individual account balances and the internal control systems that control a firm's accounting processes. However, using existing procedures it has not been possible to integrate the results of these isolated statistical measurements into consolidated measures of the reliability of either accounting balances or summary totals such as total current assets, total liabilities and total income.

This dissertation develops Bayesian mathematical procedures for constructing such models using the prior judgments of an auditor, statistical sampling results and auxiliary accounting information available from the firm being audited. The procedures allow the auditor to integrate such evidence from several distinct internal control systems and account strata into a single probabilistic measure of the auditor's uncertainty in the total dollar error in a single account or group of accounts. The procedures can be used with either a detailed decomposition of the processing steps of an internal control system or with a composite internal control system perspective.

These auditing capabilities are based in part upon several original mathematical studies presented in the appendices of the dissertation. A theory has been constructed for approximating an unknown probability density function with a series expansion of Jacobi orthogonal polynomials. It is shown how the coefficients of sequential and Edgeworth forms of the expansion can be determined from known probability movements of the unknown probability density functions. In a separate study a Poisson-gamma model is developed for specifying a probability distribution for the total error amount in a low error rate population. A joint probability density function of Bayesian natural conjugate form is developed in the course of this analysis for the skewness and scale parameters of the gamma distribution.

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The researcher has developed a conceptually sound study of the integration of auditing evidence. The motivation for and interest of the Certified Public Accounting profession in such procedures has been discussed in detail. An extensive literature review of the related auditing research has placed this study in clear perspective with respect to prior research. A survey given in the appendix of many of the properties of an extended form of the beta probability distribution is of general mathematical interest. The expositional form of the dissertation allows both mathematical and nonmathematical readers to study the analysis.

The research makes a worthwhile contribution to the theory of the integration of auditing evidence. The mathematical statistical theory developed is of general interest and of particular interest to those interested in the implications of uncertainty on aggregate summary measures.

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Date Nov. 28, 1977

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To Ute and our three children
Nauko, Dagmar and Birgit

whose sacrifices and love made
it all possible

CHAPTER 1

AN OVERVIEW OF THE DISSERTATION

1.1 Preface

In recent years the accounting profession has seen a proliferation of auditing research concerned with the methods used by Certified Public Accountants in attesting to the reliability of financial statements. A similar research trend in financial accounting has focused on the impact of the information presented in these audited reports on financial markets. Both trends may be viewed as part of a growing concern of our society for the responsible utilization of those economic resources subject to major market imperfections.

While some audited financial statements were publically available before the S.E.C. Act of 1933, today's audited financial statements represent a significant commitment to a service not allocated by market forces. Many free market imperfections can be envisioned to support our nation's regulatory posture towards financial statements. However, since our regulatory bodies and the supporting public accounting industry are not without their economic and policymaking imperfections, the implication for public policy of these free market imperfections is not clear.

It is hard to envision a good or service with as many free market imperfections as published accounting reports. Since the factor inputs necessary to produce a firm's accounting reports are generally controlled by one firm without government regulation monopoly conditions would prevail. Further, since the marginal cost of producing an additional copy of a report is often declining (and nearly zero) without regulation a natural monopoly could exist.

Economic externalities and public good characteristics may also be associated with publically available accounting reports. Many of the hypothesized benefits of these published reports, such as providing information for setting stock prices and the allocation of new investment capital, represent positive externalities of the use of the reports. These externalities also lead to potential public good characteristics of accounting reports. Since the marginal cost of providing another individual with the benefits of these hypothesized externalities is essentially zero and it is also difficult to enforce property rights and exclude individuals from benefiting from such potential externalities, public good characteristics may arise from these externalities.

Finally, it is often hypothesized that through regulation the value of the reports can be enhanced. Thus, standardization and external auditing can affect the comparability and creditability of financial statements. Again there is a lack of excludability and essentially no marginal costs for supplying these benefits to another individual. Thus, standardization and external auditing may lead to public good characteristics of accounting reports not possible in an unregulated environment.

There are two major orientations to recent academic research concerned with the regulated public accounting industry. Financial accounting research has tended to take a benefit perspective and focus on the above types of economic impacts of current and potential forms of published accounting reports. Information systems and auditing oriented research has tended to take a cost perspective and focus on the societal production process used to generate these audited accounting reports.

This dissertation is concerned with one aspect of the auditor's involvement in this production process. Namely, how the information used by the auditors to attest to the reliability of account balances is collected and consolidated. Analytical methods are developed for integrating the diverse sources of auditing evidence, pertaining to account balances, that arise in this review and testing phase of an audit. The collection and utilization of auditing evidence for possible footnote disclosures is not examined.

There are many factors that the auditor can consider in forming an assessment of the validity of a stated account balance. The general business setting of the firm and the associated economic pressures and risks are important subjective considerations. The auditor's preliminary examination of a client's information system, his formal review and testing of the client's internal control systems, and his substantive sample of an account provide a more structured basis for assessing the validity of an account balance. Also of interest to the auditor is the relationship found in a substantive sample between individual sample observations of an account and the corresponding elements of the client's records.

The presence in a firm of physically or logically distinct internal control systems can suggest a need for the auditor to stratify his evaluation process. Such distinct internal control systems (i.c.ss.)* may arise as a result of a divisional and/or subsidiary structure, or may be due to varying degrees of reliability of the firm's processing and

*The technical abbreviations, mathematical notation and numbering system used in this dissertation are defined in the first section of appendix 1.

control systems. The resultant stratification motivates a need for consolidating the individual assessments of i.c.ss. and stratified accounts. Similar requirements can arise when the auditor wishes to combine his assessment of several accounts in order to develop an aggregate total such as total current assets or net income.

Mathematical methods are developed in the dissertation for consolidating these diverse sources of evidence. These methods allow the auditor to develop an "evidential integration" model of his uncertainty about a firm's account balances. More specifically, the dissertation is concerned with how the components to the auditor's evaluation of a firm's accounting system might be structured, and how this structure can be consolidated into summary measures of account balance errors. The vehicles for this endeavor are probability density functions (p.d.fs.) for error rates, for sizes of identified errors and for the total error in one or more accounts. The dissertation is thus concerned with how the latter p.d.f. can be determined from the former p.d.fs.

The dissertation's evidential integration focus leads to the following fundamental research question:

> Can increased levels of auditing efficiency and confidence be achieved by analytically consolidating into summary measures the diverse sources of auditing evidence contributed by the auditor's formal review and testing of control systems and accounts?

This question does not focus on specific research. The question serves as a research goal toward which more specific inquiries can contribute.

In asking the question "Why evidential integration?" chapter 2 suggests an affirmative answer to this fundamental or primary research question. The discussion of this chapter proceeds with an analysis of the current judgmental process of interpreting statistical sampling evidence. The more formal procedures of an evidential integration model are then shown to suggest a step toward a more objective and relevant utilization of the auditor's informed judgment.

There are two secondary research questions that logically follow from the primary question. First, the analytical question of "How can one structure such an evidential integration process?" Second, the subsequent empirical question of "How useful (in the primary question's sense) is the resulting structure?"

The dissertation concentrates on the first of these secondary questions. Again there are two complementary aspects to this research question. First, the behavioral question of "How can the auditor's informed judgment be quantified?" and second, the analytical question of "How can a quantified form of the auditor's judgment be analytically combined with additional sources of evidential information?"

This second question is the focus of the dissertation. It is shown how such diverse information can be analytically combined into a usable model.* Without the availability of such a model building capacity there has been only limited motivation for perfecting the judgmental assessment methods of the first question. It is hoped that this dissertation will motivate subsequent behavior research to define and test in the auditing environment such judgmental assessment methods (e.g., Felix 1976).

Such behavioral research will be necessary before this dissertation's evidential integration model can be field tested. While there is

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*A case study analysis in chapter 7 illustrates this process. Readers not familiar with the auditing process may find the initial development of the case presented in sections 7.1 and 7.2 helpful in gaining a greater appreciation for the types of problems that motivate this research.

a vast literature in this area, there are still many unanswered questions. Of particular interest to this dissertation are questions related to the assessment of skewed probability judgments and the statistical correlation that may exist between individually assessed random variables. The typical use of an audit team with at least one senior CPA and several junior auditors raises additional issues related to group versus individual assessment performance.*

1.2 The Auditor's Use of an Evidential Integration Model

The goal of the dissertation is to demonstrate how a linkage can be developed between audit evidence and its implications on account balances. When these procedural tools are applied to a specific audit environment, the resulting evidential integration model can be used to explore the logical implications of the decision maker's judgment and the available statistical evidence. The procedures used in developing such a model build upon numerous Bayesian prior to posterior analyses of individual components of the audit environment. A complete model brings these components together, and acts as a linkage between input judgments and the final implications on account balances.

More specifically, an evidential integration model provides a linkage between the auditor's judgments about internal control system (i.c.s.) errors and the implied net error in an account or series of

*An extensive review by Hogarth (1975) is a useful starting point for investigating such issues. The accompanying comments and further references by Winkler and by Edwards should not be overlooked. A more specific review of research pertaining to assessment methods is given by Chesley (1975). Hogarth has discussed all the major investigations of subjective probability and cites a number of earlier reviews and anthologies. Chesley has more specific accounting objectives and reviews in greater depth a number of measurement methods.

accounts. This process involves a logical analysis of the auditor's uncertainty in his judgment about specific error rates and/or error sizes* in light of sampling evidence, the client's stated account values, and special partitions of accounts and i.c.ss. suggested by the evidential review process.

While this formal analysis may seem strange to the auditor, the underlying notion is not new to the auditor. A series of searching, self-examining questions is a critical aspect of an audit engagement. The decision maker must question the accuracy of his judgment, and ponder the implications of deviations in his judgment from the unknown truth.

In essence every audit engagement is a research project to which judgment, logic, statistical methods and electro-mechanical aids are applied. An evidential integration model draws upon the latter two tools in order to strengthen the logical basis of the research project. The astuteness and creditability of the auditor's judgment is enhanced when he can explore and objectively document the logical implications of his judgment. Without such methods the auditor may fail to fully capitalize on his judgmental review of the firm's processing and control environment. The resulting overemphasis on statistical sampling can be a source of economic inefficiency to the ultimate consumers of the firm's goods and services. Further, any procedure that increases the creditability of the auditor's attestment could possibly affect the investment risk premium that these same consumers ultimately absorb.

The auditing of a modern corporation is far too complex a task to presuppose that mathematical models currently could be used to generate

*This uncertainty is subsequently referred to as the auditor's "judgmental uncertainty."

audit decisions. Rather, the model is an experimental tool. Through the power of mathematics and computer technology the model allows the decision maker to explore the risk implications of trial judgmental inputs and potential sample evidence. Or, the model can be used to isolate those situations in which worst case input judgments lead to unacceptable final risks. Thus the model can be used to cull out those audit situations worthy of more intensive examination and to explore for these situations the implications of input judgments.

The insight gained through such an analysis is another source of evidence to be weighted at the opinion formulation stage of the audit engagement. This evidence shows the implications of subdividing the auditor's inevitable total error judgment into a greater number of more manageable component judgments. Modern business education would seem to be predicated on such a process. That is, informed judgment is inevitable in business decision making. But the risk of errors in the final decision often can be lessened by building up evidence from a multitude of lower level input judgments.

The auditor's current use of his judgment in interpreting the results of statistical sampling is discussed in chapter 3. It is suggested that what has come to be called a Bayesian perspective really is a step toward a more rational and objective use of this judgment. Independent of these conclusions, the procedures developed in this dissertation are of value even to those auditors who have serious reservations about using Bayesian judgmental analysis in the auditing process.

This observation follows from the interrelationship between classical statistics and a Bayesian analysis. The usual classical statistical analysis of sample evidence can be shown to be a special case of certain Bayesian results. It is only necessary to use the Bayesian results with a special case of indifferent or noninformative prior evidence. This fact leads to a more remarkable observation. A Bayesian analysis such as used in this dissertation at times can be used to analyze practical problems that have not yielded to classical analysis. New classical statistical solutions then can be determined by applying the noninformative prior specification of evidence to the Bayesian analysis.*

1.3 The Development of the Evidential Integration Model

Research directed at aiding the auditor in his decision making process can adopt a number of diverse focuses. Possible areas of research include the auditor's decision making processing, the evidential gathering activities of the auditor that supports this decision making process, and the methods used by the auditor to specify his uncertainty about the parameters of potential auditing models. The extant auditing statistical research literature for all of these territories are reviewed in chapter 3.

The analysis of this dissertation concentrates on the evidential gathering activities of the auditor. A possible extension into the auditor's decision making process has not been undertaken. The general form such an extension might take is illustrated by Felix and Grimlund (1976). Also, as discussed in section 1.1, the behavior issues pertaining to the specification of judgmental uncertainty are not analyzed.

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*One of the theoretical results used in this dissertation illustrates this process (Felix and Grimlund 1977). The classical degeneration of the analysis provides a potential solution to a non-Bayesian auditing sample problem recently explored by Kaplan (1973b).

The dissertation's evidential gathering focus involves two major considerations: the nature of i.c.s. evidence, and the implications of this evidence on account balances. Auditing oriented procedures for examining these two areas are developed in chapters 4, 5 and 6. These three chapters provide the necessary logic for constructing an evidential integration model. An example of the use of this logic is given in chapter 7.

While it has been found convenient to introduce the notion of an i.c.s., this is just a shorthand for a repetitive set of internal control or processing steps used to record economic events. The essential attributes that the term i.c.s. is intended to convey are repetition and statistical consistency. Thus, the a priori probability of an error having occurred in the processing of a randomly selected document through an i.c.s. is assumed to be constant.

Analytical procedures for describing such i.c.ss. have been developed by Cushing (1974) and by Yu and Neter (1973). These authors assume that the probability of error for each processing step of a document flow is known. In chapters 4 and 5 the approaches of these authors are consolidated with the further assumption that there exists a p.d.f. for each fixed but unknown processing step error rate. Probabilisitic procedures are then developed for determining a p.d.f. for the composite error rate of transactions flowing from the i.c.s. These new developments can be used by an auditor to specify and explore the implications of his uncertainty about error rates within i.c.ss.

In chapter 6 procedures are developed for combining the transaction error rates developed in chapters 4 and 5 with dollar error size information. This analysis builds upon the prior work reported by Felix and Grimlund (1977). The current analysis deals with the problems that arise when several i.c.ss., numerous transaction types and a variety of accounts all are relevant. It is shown how a p.d.f. for the total amount of error in one or more accounts can be consolidated from these diverse sources of evidence.

Chapter 7 deals with implementation. While it would in theory be possible to use the procedures of chapters 4, 5 and 6 to completely model an account environment, such an application would be highly unusual. Rather, these procedures can be used to explore sources of weaknesses in a firm's system of internal controls. An extensive example of this process is developed in chapter 7 and used to illustrate several of the procedures of the previous chapters.

1.4 Mathematical Appendices

In the second half of the dissertation, mathematical procedures are developed to accommodate the evidential integration model. This work is presented in appendices 1 through 5. With a few exceptions these appendices represent new mathematical expositions, but not necessarily all original mathematics. The subsequent summary of these appendices will clarify this distinction.

Appendix 1 collects together a number of miscellaneous procedures necessary to support the analysis. Of particular interest is the discussion of several methods for approximating the joint moments for correlated random variables (r.vs.).

In appendix 2 approximating procedures, moments, cumulants, and other properties are developed for "extended" beta p.d.fs. defined on an arbitrary interval [a,b]. These results are useful in analyzing error rates and in approximating the p.d.f. of one or more total error random variables (r.vs.). Many of these results are relatively straightforward, and undoubtedly have been derived many times. One result, the recursive cumulant relationship, has never been published in journal or book form. The published forms for many of the other results, when available, have been found to be incomplete, occasionally in error or in obscure sources. What is lacking in the literature is a convenient compendium in a common notation that brings all this material together and fills in the missing gaps. This is the objective of appendix 2.

Appendix 3 develops procedures for approximating an unknown p.d.f. with known moments by a Jacobi orthogonal polynomial expansion. Truncated forms of the expansion are shown to be a linear function of beta p.d.fs. This mathematical form resolves a major analytical problem that otherwise arises in combining i.c.s. error rate information with error size data. Appendix 3 reviews the theory of orthogonal polynomials and develops new procedures for approximating a p.d.f. using Jacobi polynomials. Beside their use in the above error rate applications, these procedures can also be used to find the p.d.f. of a linear function of "beta-normal" r.vs. This application allows the auditor to consolidate the total error uncertainty that arises from several weaknesses in separate i.c.ss.

An important analytical tool of the dissertation is the beta-normal p.d.f. of Felix and Grimlund (1977). This p.d.f. can be used to combine error rate and error size uncertainty. Appendix 4 summarizes these results and derives a number of properties of the beta-normal distribution given without proof by Felix and Grimlund. Several additional properties of the distribution are also derived and used to motivate and explore the use of an extended beta p.d.f. for approximating a beta-normal p.d.f.

Appendix 5 develops an alternative Poisson-gamma model to the beta-normal procedure of appendix 4. A natural conjugate distribution for the skewness and scale parameters of a gamma distribution process is derived as part of this development. The Poisson-gamma model is found to be not nearly as tractable as the beta-normal approach. A third alternative based on the beta and gamma p.d.fs. is discussed in paragraph 6.2.2. This alternative combines the attractive features of the beta-normal and Poisson-gamma approaches. Unfortunately, it is found to be just as cumbersome to use as the Poisson-gamma model.

1.5 Concluding Remarks

Four different classes or modules of mathematical procedures are used to develop an evidential integration model. These modules are used to determine and consolidate the auditor's uncertainty pertaining to both error rates and total error amounts. The dissertation's organization with respect to these four modules is shown in figure 1.5.1. The sections and appendices most relevant to each module is shown in the figure. The appendices of primary importance for a module are shown in the upper part of each diagram. For each module the relevant inputs and outputs also are shown. For instance, in the error amount determination module, the output is a beta-normal p.d.f. which affects at least two accounts as a result of the double entry booking structure.

An example of how these modules can be combined to develop an evidential integration model is given by figure 1.5.2. In this figure three i.c.ss. generate error rates. However, since the dollar error

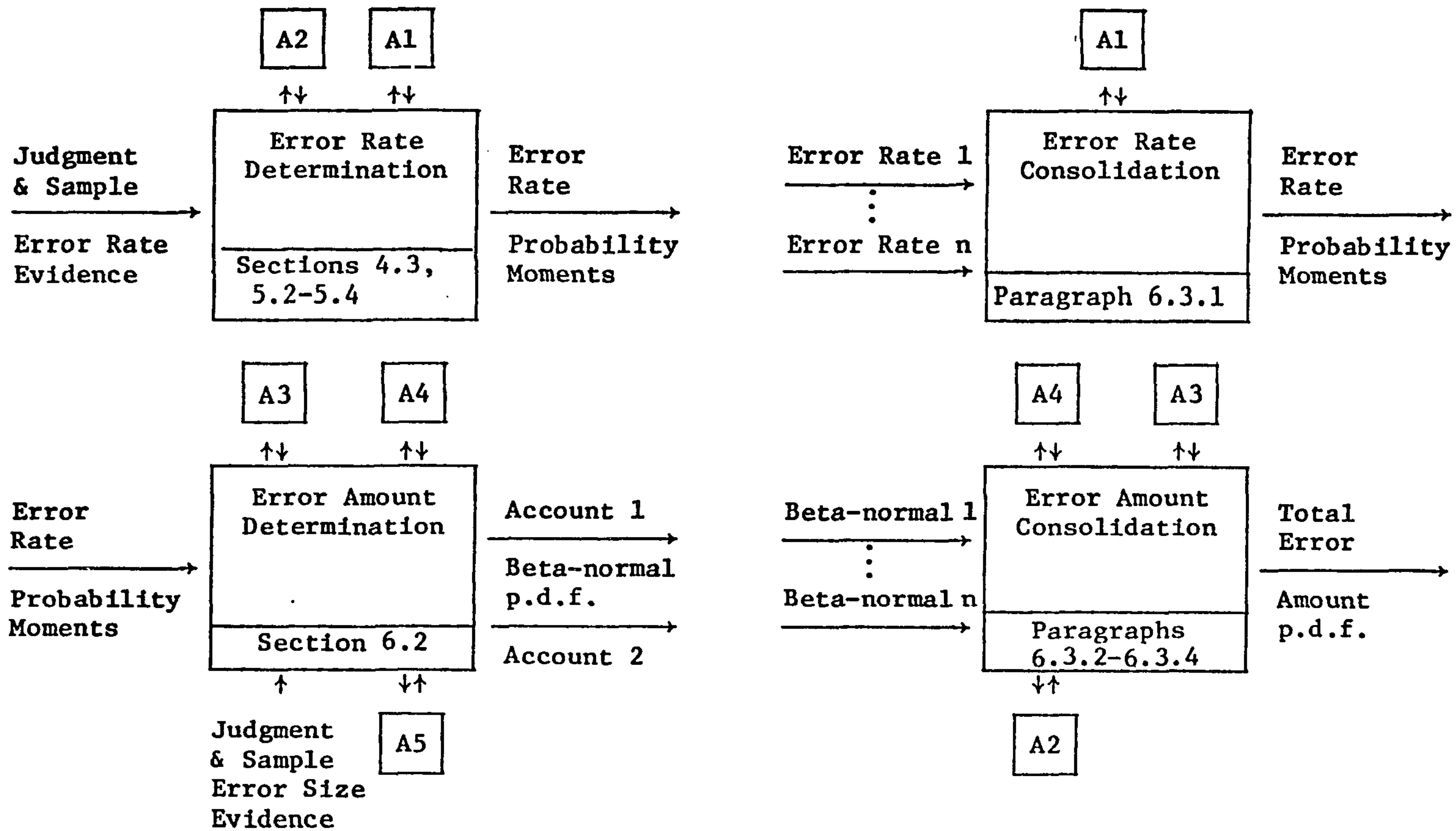


Figure 1.5.1. Evidential Integration Modules and the Dissertation's Related Discussions. Appendices are indicated by A1, A2, etc.

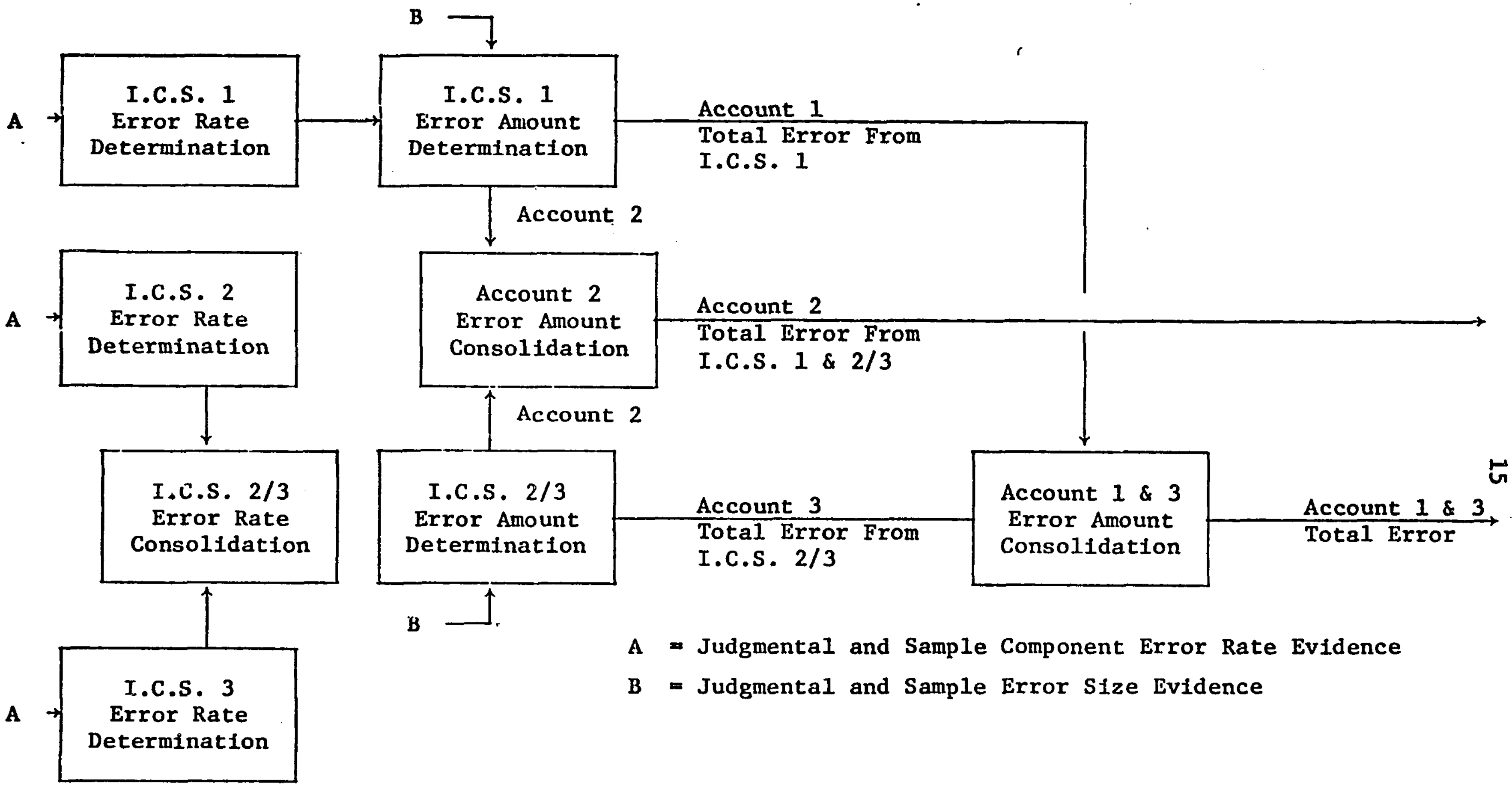


Figure 1.5.2. The Determination and Consolidation of Audit Evidence. An example of an evidential integration model.

sizes need not be distinguished between i.c.ss. 2 and 3 a preliminary error rate consolidation module is illustrated. Since account 2 collects errors from both i.c.s. 1 and consolidated i.c.s. 2/3, the total error p.d.f. for this account is found through an error amount consolidation. Finally it is assumed that accounts 1 and 3 are of the same class (i.e., current assets), and consequently a consolidated p.d.f. is desired for all errors of the class.

As is to be expected and can be seen from a close examination of figures 1.5.1 and 1.5.2, the expositional flow in the subsequent chapters of the dissertation does not strictly correspond to the logical flow illustrated in these figures. An exposition that completely parallels the logic flow would tend to prematurely address secondary issues and reduce the clarity of the discussion.

CHAPTER 2

WHY EVIDENTIAL INTEGRATION?

2.1 Preface

The application of statistical methods to auditing has become a very active issue of interest to accounting practitioners and theorists. In recent years there has been a multitude of articles, working papers and symposium discussions calling for research, presenting methodological suggestions or developing normative models. The Accountants Index provides one measure of this interest. In the last 5 years there have been approximately 165 citations listed under "Testing and Sampling."

In reviewing this literature it is apparent that the focus of interest in auditing with statistical sampling and inference has shifted. As will be indicated, the accumulative evidence today suggests that there is wide acceptance of the statistical approach to testing transactions and specific balances. Thus, the level of training among practitioners appears to be a major determinant of the degree of use of these methods.

This wide acceptance of the statistical approach has shifted the focus of research to questions of summarizing multiple sources of evidence. The long-standing research question, of how statistical procedures fit into the overall evidential gathering process, has taken on a new significance. As the profession looks ahead, one ponders if these procedures will remain singular sources of information. Or can increased levels of efficiency and confidence be achieved by formally consolidating these isolated sources of statistical evidence with standard auditing evidence in an evidential integration process.

As mentioned in section 1.1 there are two secondary dimensions to this research question. First the analytical question of how such an evidential integration process can be structured. And second the empirical question of how useful to the auditor is the resulting structure. Any potential resolution of these secondary questions must be based upon analytical and empirical research. However, before undertaking such research there is a prerequisite need for some a priori bases for hypothesizing the ultimate success of the overall endeavor. In this chapter the relevant views of practitioners and researchers are discussed in conjunction with several independent observations of the author.

2.2 Where We Stand Today

The application of statistical methodology to auditing is very understandable. Both disciplines strive to use specific observations as a basis for making evidentially supportable generalization. Such logical similarities provide normative support for the use of statistics in auditing. A more descriptive test of the operational appropriateness of the methodology can be derived from the collective appraisal of practitioners.

In recent years the use of statistical procedures by independent auditors has been surveyed by a growing list of researchers including Jacobs (1971), Dennis (1972), Joseph (1972), Ross, Hoyt and Shaw (1972), Hubbard and Strawser (1972,1973), Barkman (1974), and Bedingfield (1975). Hubbard and Strawser's (1972) sample indicates that "CPA have made substantial progress in integrating statistical sampling methods in their practice in recent years" (p. 673). Through in-depth interviews with personnel of the largest accounting firms, Barkman found that the use of quantitative procedures by these firms was restricted to classical survey sampling and hypothesis testing.

Bedingfield's recent survey suggests that CPA partners in the aggregate are favorably disposed to such methodology. Bedingfield concluded that: "Statistical sampling is widely used in the test of transaction (i.e., compliance tests) and in those areas of balance verification (i.e., substantive tests) characterized by a large volume of defined sample units . . ." (p. 54).

This general interest in statistical sampling by auditors is not limited to independent CPA firms. In discussing the use of such procedures by the U.S. General Accounting Office, Gentile (1974) notes that "three professional staff statisticians have received over 850 requests for assistance in the past year from the audit staffs" (p. 16). Gentile's concluding paragraph is of particular interest.

> Today, more than ever, there is a pressing, almost desperate need for reliable data which can be used for decisionmaking in every field. This need will continue to grow, and auditors will be asked to provide an increasing share of it. Because statistical sampling is the only practical way of obtaining data of known reliability, it is a practical necessity for today's auditor to have a working knowledge of it. (p. 16)

This need for training is a recurring theme in the survey literature. Bedingfield observed that the "two most prominent reasons (given by CPA partners) for not using statistical sampling were the lack of training and the feeling that statistical sampling is not as relevant for firms that service mainly small clients" (p. 53). Kinney and Ritts (1973, pp. 3-4) cite several other authors including Ross, Hoyt and Shaw, who have reached similar conclusions. Joseph (1972) commented in his abstract that "offices of large firms expressed their apprehension about accepting the use of statistical sampling, because of the disadvantages when placed in the hands of inadequately trained personnel."

A search for objectivity appears to be the driving force drawing auditors to statistical sampling. Bedingfield states:

> The most common reason offered for initially resorting to statistical sampling was the objectivity of the technique. This objectivity is evidenced in two respects--the objectivity of the technique when viewed by third parties (i.e., it is defensible) and the objectivity inherent in the analysis of the results. (p. 54)

Some indication of what auditors may perceive as objectivity to a third party can be drawn from the following remarks by Stoker (1971) of Haskins and Sells:

> . . . I felt that a great benefit that we derive from the use of statistical sampling is the increased quality of audit work of our staff accountants, particularly in establishing the direction of the audit test and definition of accounting populations. (p. 138)

In discussing "What are the courts saying to auditors?" Sommer (1972) summarizes the rising third party judicial pressures that suggest a greater need for such objectivity:

> . . . the performance of the accounting profession is going to be increasingly subject to judicial scrutiny. As the task of bringing class suits has been moderated, and as potential plantiffs and their counsel have witnessed the ease with which judicial intervention may be secured, accountants increasingly may expect to have their work thrust into the judicial arena either by private litigants or the S.E.C. (p. 33)

Current empirical evidence provides no basis for hypothesizing that the current trend toward statistical sampling will result in increased procedural efficiency. The CPAs surveyed by Bedingfield expressed mixed feelings about the potential for cost reductions with statistical sampling. Bedingfield does not draw any statistically significant conclusions from these results. In a field experiment Kinney and Ritts (1973) developed evidence suggesting that the statistical sample size of practicing auditors are not smaller than the equivalent judgmental sample size. These descriptive results contrast with the normative potential for smaller sample sizes found by Aly and Duboff (1971).

The early experience of Haskins and Sells also is indicative of an objectivity rather than efficiency effect. Stringer (1963) noted that for nearly 400 applications of the Haskins and Sells sample plan spread over 150 engagements that:

> The effect of this plan on audit time was not significant . . . These results confirmed the premise on which the plan was originally developed and advocated--namely that it would provide a more objective basis for audit sampling but would not necessarily reduce audit time. (p. 411)

The growing use of statistical sampling is not surprising in view of the implicit emphasis on objectivity of the third standard of field work of the AICPA Statement on Auditing Standards (S.A.S.) No. 1 (1973):

> Sufficient competent evidential matter is to be obtained through inspection, observation, inquiries and confirmation to afford a reasonable basis for an opinion regarding the financial statements under examination. (§ 150.02)

Statistical sampling with its explicit statement of sampling precision and reliability can, when properly exercised, conform to the spirit of the standard.

Objectivity is of course an admirable goal for auditing, but a caveat is in order. As succinctly stated by Broderick (1974) (an Arthur Young parnter), the "exercise of judgment is at the heart of auditing" p. 77. Thus in this view the professional auditor contributes more to the auditing process than merely to act as a concert master for a collection of objective procedures.

The third S.A.S. standard gives explicit recognition to the opinion or judgmental nature of auditing. S.A.S. No. 1, section 320A, further explores the interrelationship between statistical sampling and auditing judgment. Section 320A.03 quotes a special report issued in 1962 by the Committee on Statistical Sampling. Of particular interest is the following passage used as an introductory statement for the subsequent discussion.

> Although statistical sampling furnishes the auditor with a measure of precision and reliability, statistical techniques do not define for the auditor the values of each required to provide audit satisfaction.
>
> Specification of the precision and reliability necessary in a given test is an auditing function and must be based upon judgment in the same way as is the decision as to audit satisfaction required when statistical sampling is not used.

The subsequent evolutionary elaboration of this basic 1962 position has been chronologized by Stringer (1972). The current S.A.S. position has amplified the judgmental relationship between compliance tests of internal control systems and substantive tests of account balances. However, the philosophical basis has not changed:

> The relative weight to be given to the respective sources of reliance and accordingly, the sampling reliability desired for his tests of details are matters for the auditor's judgment in the circumstances. The committee believes that reliability levels used in sampling applications in other fields are not necessarily relevant in determining appropriate levels for applications in auditing because the auditor's reliance on sampling is augmented by other sources of reliance that may not be available in other fields. (§ 320A.15)

The current guidance in these matters provided by S.A.S. No. 1 recently was summarized by Taylor (1974):

> The auditor should first make a preliminary decision about the internal control over a particular class of transactions or balances. If he decides that he cannot rely on the internal control to prevent material errors, then he should obtain his satisfaction solely from substantive tests. As a result, it would not be necessary to test for compliance deviations since no reliance is being placed on the internal control.
>
> If the auditor does make the preliminary judgment that the internal control can be relied on to prevent material errors, he will be able to adjust downward the reliability level he will need for related substantive tests. To support this reduced reliability, the auditor must satisfy himself that his preliminary decision on internal control was correct. (p. 80)

A number of suggestions have been made on how to implement such general guidelines. Elliott and Rogers (1972) have developed a substantive hypothesis testing strategy based upon the auditor's judgment of the internal control systems. An algorithm is used to establish the risk levels of the tests. Smith (1972) has commented on the confusion of several authors about the relationship between reliability and precision. Smith also contrasts Bayesian judgmental assessments with reliability and precision judgmental assessments. Broderick (1974) has examined in some depth the auditing implications of judgmental reliability and precision assessments. He presents tables for translating judgments such as "excellent," "fair" and "weak" into numerical reliability and precision values.

2.3 Looking Ahead

In looking towards the future several questions are apparent. Can the current reliability and precision judgmental assessment process be improved upon? Will the current techniques serve the needs of the future socioeconomic/legalistic environment?

One appraisal pertaining to the first question has been given by Yu and Neter (1973):

> Statistical sampling is one means of obtaining objective audit evidence. Unfortunately, the use of this tool has been fragmental, and its potentiality for reducing subjective elements of the auditor's judgments on internal controls and the evaluation of account balances has not been fully exploited. (p. 274)

Looking at the second question Loebbecke (1974) commented that "if we are to provide a high level of audit services on a continuing basis, we must use techniques to preserve objectivity" (p. 73). In examining means of achieving objectivity, Loebbecke's thinking adds analytical depth to the previously examined broad interest among auditors in objectivity. Summarizing several advantages to a decision model presented by Felix (1974), Loebbecke sees objectivity as being achieved through:

1. Control of risk through precise definitions
2. Expression of decision criteria in more meaningful terms
3. A vehicle to motivate better response to changes in the audit environment
4. A framework for improved communication both between auditors and with those affected by auditor results. (p. 73)

Of particular interest in evaluating the current judgmental reliability and precision assessment process is Loebbecke's first objectivity criterion (originally mentioned by Roberts 1974, p. 48). When the auditor determines sample reliability and precision levels, and then states that because of additional evidence the situation is more favorable, there is no precise definition or control of risk. This is not to say that the auditor's assertion is not totally accurate, only that there is no basis or standard on which a third party can evaluate the risks.

An AICPA case study on the "Extent of Audit Samples" gives some ground for concern about how a third party can evaluate these risks.

Stoker (1971) of Haskins and Sells, in discussing the case study, concluded that the variation in judgment among eight auditors out of separate firms "indicates some inconsistency in judgment" (p. 137) among the auditor's appraisal of the case.

Such inconsistency is hardly surprising. The auditor is being asked to take his directly observable data pertaining to internal control error rates and dollar error amounts, and translate them subjectively into something which he has little experience with. Whether the auditor deals with sample sizes or desired reliability and precision levels, the process is the same. There is no link or feedback which relates the observed data with the demands the auditor is called upon to fulfill. This is a missing or weak link in the current judgmental procedures of auditors.

The present circumstances clearly violate Carmichael's (1972) concept of a comprehensive auditing theory:

> A theory of auditing should be an organized and systematized body of knowledge of the field of auditing, which identifies the variables of the auditing practice and explains their importance, interrelationship, and implications. (p. 102)

In particular, the potential for understanding the interrelationship and implications of auditing judgmental variables would seem to be enhanced if the auditor's judgment focused on his observable data. By focusing on error rates and error amounts the auditor could apply his inevitable and valuable judgment to attributes that fall within his realm of experience.

In order to continue the logical trend of this analysis, the concept of Bayesian judgment can be introduced. There are currently, of course, analytical problems in collecting and bringing together Bayesian judgments over a multitude of i.c.ss. Aside from these technical issues, some auditors are apprehensive about introducing the "subjectivity" of a Bayesian process.

While perhaps a valid concern in some disciplines, this concern is not the real issue in auditing. The subjectivity of the auditor's judgment is an inevitable aspect of the evidential gathering process. Abstracting somewhat, the current evidential auditing model is already a Bayesian process. The real question is an empirical question that can be dealt with through research. Can the auditing process be improved by directing the auditor's judgment at the observable error attributes, rather than by relying upon a judgment at higher levels of summarization? Succinctly stated, is the auditor's judgment more informative at a lower level of abstraction?

A priori considerations suggest an affirmative response. Judgments based upon lower levels of abstractions may relate more directly to the auditor's past experience. In addition, much of the scientific development in business and in other disciplines has focused on pushing back the level of measurement to a lesser and lesser degree of abstraction. Thus, this question focuses in part on whether or not this historical trend is also applicable to auditing.

Stringer (1972) of Haskins and Sells adds weight to such an affirmative a priori conviction. In considering statistical sampling standards he noted that:

> Any presentation of a mathematical model in which subjective judgments and objective measurements are combined invites the somewhat annoying, but nevertheless completely accurate, criticism that the former cannot be quanitifed precisely. This criticism, however, does not impugn the usefulness of a model in focusing attention on the separate elements in a

> complex problem, and in showing the relationship between these elements. Furthermore, this criticism invites the rebuttal that it is more rational to quantify some of the separate elements of a problem subjectively if necessary, than to deal subjectively with the entire set of elements where some can be measured objectively. (p. 49)

In their examination of the third party legal ramifications of auditing, Reiling and Taussig (1970) also have expressed interest in a Bayesian approach.

> Financial reporting would be greatly improved if a Bayesian probability approach were applied to the financial statements. . . . Many lawsuits would be avoided if auditors would simply indicate that they are not certifying to deterministic facts, but rather expressing an opinion on estimates from probability distributions. (p. 45)

As is often the case in the development of new methodological approaches, Bayesian auditing techniques have been proposed on a fragmented basis. While necessary in order to awaken research interest in more meaningful composite procedures, such early proposals also create disinterest because of their narrow focuses within a much larger problem.

2.4 Concluding Remarks

The procedures developed in this dissertation for constructing an evidential integration model can significantly expand the scope of such Bayesian analyses. The total implication of tests of both i.c.ss. and account balances for a multitude of systems and accounts can be considered. An evidential integration model can be used to develop probabilistic statements of the total error in financial statement accounts. Consequently, it might be possible to implement the previously quoted suggestion of Reiling and Taussig. Also, such a statement of the probabilistic nature of a financial statement can increase the communication between the auditor and those affected by the auditor's results.

Finally, by reducing the level of abstraction, that is used by the auditor in making judgments, the process of setting professionwide guidelines might be enhanced. As a previous quote from Sommer has indicated, such standards could have real legal and hence economic significance to the auditor. By allowing the auditor some measure of risk protection, standards help to avoid overly conservative auditing procedures. Auditors, clients and society in general tend to benefit from the avoidance of such extreme legal pressure.

CHAPTER 3

A REVIEW OF THE LITERATURE ON STATISTICAL PROCEDURES IN AUDITING

3.1 Preface

The literature of auditing statistical methodology and related research presents a very confusing multitude of procedures and proposals. There is a vast array of assumptions and results only loosely related. In order to understand what has been accomplished, what remains to be achieved, and what this dissertation does, it is convenient to first develop a conceptual framework for classifying auditing statistical research and methodology.

The major emphasis of auditing statistical literature is on compiling information and presenting it in a quantitative framework. One significant difference is the type of information that is utilized. The conceptual framework presented in the next section focuses on the variety of evidence utilized and the degree (or level) of evidential integration. This evidential integration framework is then used in section 3.3 as a basis for reviewing the auditing statistical literature, and for focusing on the methodological gaps that are considered in this dissertation. In order to serve this latter objective, a pyramiding structure of successively higher levels of evidential utilization and integration has been used. This evidential ranking structure does not in any way measure the relative utility of the accompanying statistical methods.

3.2 An Evidential Integration Framework for Classifying the Literature

At a lowest level of evidential integration sample data are extracted and summarized without matching up each sample observation with the client's corresponding stated value. Only, aggregate sample measures are compared with the corresponding summary values from the client's records. Mean-per-unit sampling and ratio estimating are two examples of this lowest level of evidential integration.

In mean-per-unit sampling this comparison between sample and client data is made at the confidence interval or hypothesis testing phase of the analysis. In using the standard form of the ratio estimating procedure only the client's stated values for the total account and the total of all sampling items are incorporated into the estimating procedure. With all these procedures the detailed control-level elements of the client's records are only used as contributory elements of aggregate summary measures.

At a second level of evidential integration each sample item is compared or otherwise mathematically combined with the corresponding client value. A substantive test with a regression estimator fits in this category. Ratio estimating procedures also are available at this level of evidential integration (Cochran 1963, pp. 176-177). Compliance testing (i.e., a document error rate analysis) is a second level "qualitative" evidential integration procedure. The acceptable processing procedure for each sample item is compared with the client's actual processing. The correct/incorrect binary classification is determined at the item level. In such second level procedures each individual sample item is compared to the corresponding item of the client's detailed records. An aggregation of all of those comparisons is then evidential integrated with the client's summary records.

A Bayesian statistical analysis of compliance errors is an example of a third level of qualitative evidential integration. In addition to the sample and client sources of evidence, a quantification of the auditor's judgmental uncertainty provides a third source of evidence. Bayesian regression estimation in substantive testing is an example of a third level of quantitative evidential integration. In these procedures there are three sources of information available for evidential integration with the client's summary records.

At a third level of evidential integration either a Bayesian compliance procedure or a Bayesian substantive procedure (i.e., an account balance confirmation) are evidentially integrated. A quantitative integration of these two procedures will be considered a fourth level of evidential integration. Detailed client compliance information, detailed client substantive information, auditing sample information and the auditor's informed judgment provide four conceptual sources of evidence available for evidential integration.

In many cases the auditor may wish to combine evidentially integrated information from several accounts (or strata within the same account). Such a procedure will be considered as a fifth level of evidential information. The previous four conceptual sources of evidence are being integrated over a multi-account environment. Such an analysis would provide composite statistical measures for the sum of these accounts (or strata).

If this composite measure is utilized in a decision theoretic expected value (or utility) analysis, a sixth level of evidential integration arises. The possible loss function implications of the auditor's decision is combined mathematically with the additional sources of evidence.

The above evidential integration pyramiding structure has introduced six conceptual sources of evidence that potentially are of use to the auditor: the auditor's sample evidence, the client's detailed compliance records, the client's detailed account records, the auditor's informed judgment, the multiple accounts and the auditor's appraisal of the loss function implication of his action. Statistical procedures that utilize such a broad range of evidence are not necessarily more useful to the auditor. The degree of evidential integration is a measure of the scope of the analysis, not necessarily the statistical power or usefulness of the analysis.

The pyramiding structure presented in this section is, of course, arbitrary. Other researchers may choose to classify various types of evidence in a different order or structure. The six types of evidence and the pyramiding structure does provide a convenient basis for understanding the scope and framework of existing research and methodology. The structure also provides a convenient basis for contrasting the dissertation's evidential integration framework with the current literature.

3.3 A Review of the Literature

Considerable statistical guidance is available to the auditor to assist him in performing isolated statistical tests of accounting records at the primary level of evidential integration. Cyert and Davidson (1962), Arkin (1963) and the AICPA (1967) series have restated in auditing terminology many classical statistical sampling procedures. In recent years several authors have suggested that such routine application of classical statistical theory to the audit environment may lead to difficulties.

Ijiri and Kaplan (1971) have suggested that the "representative" classical sampling objective may be too restrictive for auditing. A subsequent survey by Hubbard and Strawser (1972) compared the multiple objective criteria of Ijiri and Kaplan with those of CPAs. Elliot and Rogers (1972) have suggested that, given the auditor's concern for Type II error, he should concentrate on hypothesis testing of accounts rather than establishing confidence intervals. They propose a nondecision theory approach for setting the probability of Type I and Type II errors.

McCray (1973) has illustrated the use of standard "level one" ratio estimation and "level two" difference estimation. Kaplan (1973b) has considered the validity of the usual statistical approximation used in such tests. He has demonstrated that significant statistical difficulties can arise when the auditor uses either "level one" or "level two" ratio (or "auxiliary") estimating procedures in such substantive tests. In the auditor's usually low substantive error rate environment, the classical statistic's t-distribution approximation of these procedures breaks down.

In a separate study Kaplan (1973a) has developed mean, standard deviation and book-to-actual correlation formulas for a particular model of the low error rate audit environment. Kaplan showed how a substantive sample size for ratio and regression estimation can be developed. He recognized that the applicability of his theory is restricted because of the fore mentioned limitations on the t-distribution approximation.

Several authors (Meikle 1972; Anderson and Teitlebaum 1973; Teitlebaum 1973; Goodfellow, Loebbecke and Neter 1974; Teitlebaum and Robinson 1975; Teitlebaum, Leslie and Anderson 1975; Kaplan 1975a; and Garstka 1976) have discussed dollar unit sampling as a potential alternative to substantive methods that rely on the questionable t-distribution approximation. While a procedure equivalent to dollar unit sampling has been utilized in the Haskins and Sells AUDITAPE system for a number of years, only recently has an open discussion of the method emerged.

Dollar unit sampling is an application of probability proportional to size sampling to a sample population composed of account balances. It is a "level two" procedure that analyzes the errors in an account at the sample dollar level. There are two aspects to the procedure: a sampling population made up of dollar units rather than control accounts, and a worst case approximation of "risk" that avoids using the t-distribution.

The substantive testing aspects of the dissertation's evidential integration model provide another potential alternative to relying on the t-distribution approximation (Felix and Grimlund 1977). This procedure can be utilized with either a dollar unit sampling frame or a control account sampling frame. Unlike dollar unit sampling the procedure does not focus on a worst case approximation of risk. Rather, the procedure expresses the total dollar error uncertainty with a probability distribution that takes into consideration the special nature of the audit environment. The procedure has been developed at a Bayesian "third level" of evidential integration. However, the procedure can be stripped of its reliance on the auditor's judgment by using a diffuse or noninformative prior judgment.

Bayesian evidential integration currently appears to be within the realm of auditing research rather than an applicable methodology. Surveys conducted by Jacobs (1971), Dennis (1972), Ross, Hoyt and Shaw (1972), and Bedingfield (1975) provide no evidence of any utilization of a Bayesian level of evidential integration. Kraft (1968) and Tracy (1969) have illustrated discrete Bayesian calculations, for compliance error rate testing. Smith (1972) has compared the sample size implications of these methods with the AICPA supported variable confidence level approach. Francisco (1972) has reviewed and illustrated a comparable procedure with a continuous probability distribution.

The performance of auditors in specifying error rate distributions has been studied by Corless (1972) and Felix (1976). Corless' experimental results show considerable variation in the judgmental responses of auditors. After a preliminary training session Felix found somewhat less dispersion in judgmental response between a quartile measurement method (also used by Corless) and an equivalent prior sample method. These results suggest the need for a well developed training procedure, and a careful consideration of the type of methods used.

The analysis of Felix and Grimlund (1977) is an example of a "third level" of evidential integration. There are no other published suggestions for quantitative level three Bayesian evidential integration within a substantive test of account balances. The sample data collected in such substantive tests cannot be realistically analyzed using standard Bayesian mathematical procedures. Since the majority of the sampled subsidiary accounts in a substantive test will be correct a sample of monetary error amounts is primarily composed of zero amounts with only an occasional nonzero item. This cluster of zero amounts violates the usual assumption of normality. The Felix and Grimlund analysis shows how these difficulties can be avoided when using detailed (i.e., item level) book value data from the client's records.

Several other Bayesian studies have avoided these "level three" substantive difficulties by not considering the availability of detailed client book values. Knoblett (1963,1970) has demonstrated how a Bayesian prior judgment could be added to a "level one" mean-per-unit classical analysis of an account balance. Deakin and Granof (1974) have shown how conditional probabilities of hypothesis testing errors can be derived from regression estimates of account balances (see also Kinney and Bailey 1976, for a fuller discussion of the underlying problems and assumptions). Deakin and Granof use these conditional probabilities to revise subjective prior probabilities of these same errors. The Bayesian analysis is used to determine a sample size for a mean-per-unit substantive test.

Using this same mean-per-unit Bayesian approach, Kinney (1974) has applied decision theory to a two state/two action formulation of a substantive test of an account. Assuming linear sampling costs and fixed Type I and Type II error losses, he showed how the auditor can make optimal expected value decisions. The procedure can be used to make sampling decisions, to determine acceptance and rejection regions, and to determine the corresponding probabilities of Type I and II errors. The procedure is also applicable to a constrained fixed sample size analysis.

In a subsequent paper Kinney (1975) extended the analysis to include a compliance test of a single i.c.s. For this single account/ single i.c.s. environment, compliance and substantive sample sizes and other optimal actions are determined. As emphasized by Kinney, the i.c.s. linkage to an account balance needs to be investigated in more depth.

Apparently, little attention has been directed toward the problem of quantitatively integrating for a single account both the indirect compliance tests of the supporting i.c.ss. and the direct substantive test of the balance. Except for a few brief observations by Cushing (1974), Kinney's (1975) procedure stands alone in the literature. The evidential integration procedures of chapter 6 link together both types of tests over a multitude of i.c.ss. These linkage procedures are based upon the audit oriented analysis of i.c.ss. developed in chapters 4 and 5.

The dissertation's "linkage" analysis leads to a probability distribution for the total dollar error in an account or a partitioned stratum of an account such as all receivables under the control of a particular division. Procedures are discussed in chapter 6 for consolidating dollar error probability distributions from several accounts (or account strata) into a single distribution. This analysis provides a fifth level of evidential integration.

When a total error probability distribution can be specified for an account (or series of accounts) a decision theory expected value analysis can be easily constructed. Sorensen (1969) has illustrated an expected net sample gain and discrete Bayesian probability revision analysis for compliance testing. For substantive testing Felix (1974) has shown how the expected net sample gain and optimal sample size can be determined from existing discrete probabilities. The two decision theory loss function procedures by Kinney have already been discussed.

Theoretical treatises exploring various loss function objectives for auditors have been advanced by Demski and Swieringa (1974), and Scott (1975a,1975b,1976).

Scott (1973) has developed a multiple accounts Bayesian decision theoretic analysis. The model is major theoretical accomplishment; however, it has several difficulties. The model utilizes nonlinear loss functions that do not flatten out at large error values. Further, the multivariate sample unit represents the total daily net error for a set of accounts. Such a sample unit is suggestive of a normal distribution process. However, for those enterprises with sufficient transaction volume to validate such a multivariate central limit theorem assumption, it probably would be economically impossible to analyze more than a few days of accounting work. The sample evidence of such a small sample size usually would not significantly alter the auditor's prior judgmental uncertainty.

3.4 Concluding Remarks

Part of the research discussed in this literature review has been summarized by major focuses of consideration in table 3.4.1. In particular, this table focuses on research concerned with judgmental specification and/or the integration of multiple sources of evidence. The scope of the dissertation is shown on the final line of the table.

It is apparent from this literature review, that the auditor is faced with a very difficult statistical problem. This is a problem that presses upon the frontiers of mathematical statistics. A combination of multiple sources of evidence, low error rates, variability and skewness in the distribution of control account balances, and the indirect consequences of actions all create an environment in which it is difficult to develop all-inclusive procedures.

Table 3.4.1

The Type of Evidence Considered in the Literature on Statistical Procedures in Auditing

(ICS) Internal Control Systems				Account Balances				Loss Data			
Detailed Decomposition	Judgmental Assessment	Compliance Testing Data	Multiple ISC's	Substantive Testing Data	Judgmental Assessment	"Client-Actual" Sample Data	Multiple Accounts (Strata)	ICS Loss Analysis	Account Loss Analysis	Authors by Subject Area and Date	Title
										Detailed ICS Decomposition Analysis	
X	X		X					X		Cushing (1974)	A Mathematical Approach to the Analysis and Design of Internal Control Systems
X	X		X							Yu and Neter (1973)	A Stochastic Model of the Internal Control System
										Standard Statistical Analyses	
		X		X						Cyert and Davidson (1962)	Statistical Sampling for Accounting Information
		X		X			X			Arkin (1963)	Handbook of Sampling for Auditing and Accounting
										Judgmental Assessment in Compliance Testing	
	X									Corless (1972)	Assessing Prior Distributions for Applying Bayesian Statistics in Auditing
	X									Felix (1976)	Evidence on Alternative Means of Assessing Prior Probability Distributions for Auditing Decision-Making
	X	X								Francisco (1972)	The Application of Bayesian Statistics to Auditing: Discrete versus Continuous Prior Distributions
	X	X								Kraft (1968)	Statistical Sampling for Auditors: A New Look
	X	X								Smith (1972)	The Relationship of Internal Control Evaluation and Audit Sample Size
	X	X						X		Sorensen (1969)	Bayesian Analysis in Auditing
	X	X						X		Tracy (1969)	Bayesian Statistical Methods in Auditing
										Judgmental Assessment in Substantive Testing	
				X	X				X	Deakin and Granof (1974)	Regression Analysis as a Means of Determining Audit Sample Size
				X	X				X	Felix (1974)	A Decision Theory View of Auditing
				X	X	X				Felix and Grimlund (1977)	A Model for the Decomposition of Audit Evidence
									X	Felix and Grimlund (1976)	Payoff Functions for Auditors: A Descriptive Analysis
				X	X					Ijiri and Kaplan (1971)	A Model for Integrating Sampling Objectives in Auditing
				X	X				X	Kinney (1974)	A Decision Theory Approach to the Sampling Problem in Auditing
				X	X					Knoblet (1970)	The Applicability of Bayesian Statistics in Auditing
				X	X	X	X		X	Scott (1973)	A Bayesian Approach to Asset Valuation and Audit Size
										Dollar Unit Sampling	
				X		X				Anderson and Teitlebaum (1973)	Dollar-Unit Sampling: A Solution to the Sampling Dilemma
				X		X				Kaplan (1973b)	Statistical Sampling in Auditing with Auxiliary Information Estimators
				X		X				Meike (1972)	Statistical Sampling in an Audit Context
				X	X	X				Teitlebaum (1973)	Dollar-Unit Sampling in Auditing
				X		X				Teitlebaum et al. (1975)	An Analysis of Recent Commentary on Dollar-Unit Sampling in Auditing
				X		X				Teitlebaum and Robinson (1975)	The Real Risks in Audit Sampling
				X		X				Gastka (1976)	Computing Upper Error Limits in Dollar Unit Sampling
				X		X				Kaplan (1975a)	Sample Size Computation for Dollar Unit Sammpling
										Integrative ICS/Account Analyses	
	X	X		X	X				X	Kinney (1975)	Decision Theory Aspects of Internal Control Systems Design/Compliance and Substantive Tests
X	X	X	X	X	X	X	X			This Dissertation	A Framework for the Integration of Auditing Evidence

CHAPTER 4

AN AUDITING MODEL OF AN INTERNAL CONTROL SYSTEM

4.1 Preface

As emphasized by the AICPA second standard of field work (AICPA S.A.S. No. 1, 1973, §320) the auditor's study and evaluation of the internal controls of an accounting system is a necessary part of an audit engagement. The specific interrelationship of these and other field procedures have been diagrammatically illustrated by Kinney (1975). This analysis shows that all routes to an opinion about a firm's financial statements require, under S.A.S. No. 1, an evaluation of the design of the internal controls of the firm.

The next two chapters are concerned with the auditor's study and evaluation of the various systems of internal control utilized by a firm. This chapter focuses on the background and implementation issues of a potential methodology for studying i.c.ss. The mathematics of the methodology is presented in the next chapter. Procedures are developed in this analysis for modeling the auditor's uncertainty about the reliability of each processing step of an i.c.s. It is then shown how these individual evaluations can be combined into a global evaluation of the uncertainty about the reliability of the journal entries that are generated by an i.c.s.

The procedures presented in these two chapters can be used to explore and summarize the auditor's judgmental and sample evidence about the operation of an i.c.s. While normally the procedures would only be used to explore areas of weakness with an i.c.s. (Moriarity 1975, pp. 32-33), the logic developed could be used to model a complete i.c.s. This logic corresponds to the error rate determination module of figure 1.5.1.

Recent articles by Yu and Neter (1973) and Cushing (1974) have outlined probabilistic functions for evaluating i.c.ss. Both approaches view an internal control processing system for accounting documents as made up of series of processing steps each of which contains several possible error rates. The two approaches differ in the type of processing steps that are analyzed. Yu and Neter focused on the flow of documents through an i.c.s. Cushing is primarily concerned with the detection of errors in documents and the procedures used to correct these errors.

In both analyses it is assumed that there is perfect knowledge of the various probabilities of error within each processing step of the i.c.s. Yu and Neter see their analysis as providing for greater objectivity in the study of i.c.ss. than is possible with the traditional use of checklists, flowcharts and questionnaires (see Brown 1962; AICPA S.A.P. No. 54 1972). Cushing states that his approach is "entirely consistent with the spirit of a Statement on Auditing Procedures No. 54" (p. 25).

The assumption of perfect knowledge of error rates* seriously limits the use of the Yu and Neter, and Cushing "accounting functions" in an audit environment. A good part of the investigative stage of an audit is implicitly concerned with evaluating the error rates assumed by these authors to be known values. Thus, procedures that recognize the auditor's unavoidable uncertainty about processing step error rates would be of

*The terminology "error probabilities" is more accurate. However, this leads in the subsequent discussion to confusing references to the probability distribution of the error probability. For this reason term error rate(s) is usually used in the following development.

more value to the auditor. Such "auditing functions" are developed in this and the next chapter.

This chapter considers several background and implementation issues that can arise in utilizing and evaluating the auditing functions developed in the next chapter. The notation of Yu and Neter is introduced. The various assumptions of statistical independence are discussed. The relationship between the auditor's traditional forms of compliance evidence and the monetary error rate emphasis of the auditing functions is analyzed. And finally the types of p.d.fs. and calculating techniques that might be used to represent the auditor's error rate uncertainty are reviewed.

In chapter 5 the mathematics of the auditing functions are developed. Then in chapter 6 these i.c.s. functions are integrated with substantive evidence from account balances. Finally in chapter 7 the application of these functions to current auditing practice is illustrated with a case study.

4.2 Background Considerations

This section introduces a number of preliminary issues, that lead up to the mathematical analysis of the subsequent chapter. The notation of Yu and Neter is introduced. In a related appendix it is demonstrated that the work of Cushing can be recasted into the Yu and Neter notational framework. This suggests that an expanded set of accounting functions drawn from both approaches with some additional embellishments can be used as a basis for modeling many i.c.ss. The various assumptions of statistical independence are next discussed. This is followed by a brief consideration of how Monte Carlo simulation might be used to derive some of the functional forms that are developed analytically in the next chapter.

4.2.1 The Use of the Yu and Neter Notation to Develop a Combined Set of Accounting Procedures

Yu and Neter developed a matrix notation for describing and modifying the probability of the various possible error states of documents as they are processed through an i.c.s. The probability for each state is represented as a state probability vector, such as (p_1, p_2, p_3, p_4). The effect of a processing step is described using a transitional probability matrix $A = (P_{ij})$ $i,j = 1,4$, where p_{ij} is the conditional probability of state j given that the document is currently in state i. Each probability p_{ij} is a component error rate probability for a processing step. Using this matrix notation the ex post (to processing) state vector is given by

$$(p_1 \; p_2 \; p_3 \; p_4) \begin{pmatrix} p_{11} & \cdots & p_{14} \\ \vdots & & \vdots \\ p_{41} & \cdots & p_{44} \end{pmatrix} \tag{1}$$

Yu and Neter illustrated their notation for several different types of document processing steps using four states: error free, nondollar errors, dollar errors and both types of errors. Thus, a document might be in an error free state. Or, it might contain a nondollar compliance error such as a missing verification of price/quantity extensions which are in fact correct. Or, it might contain a dollar error that will affect account balances. Finally, the document could be in the fourth state with both types of errors.

Yu and Neter's four state partition is one possible approach for including in the analysis information about nondollar compliance errors. Auditors generally find this information useful in making appraisals about possibly accompanying dollar errors. A two state (error free, dollar error) version of the Yu and Neter notation is used in this dissertation in conjunction with a special analysis of the relationship between dollar errors and nondollar compliance errors.

Cushing examined in greater detail the error control aspects of an i.c.s. under the assumption of two possible states: error free and in error. Yu and Neter assumed that all documents singled out as potentially in error are correctly analyzed. Cushing assumed that new errors could be created within the error control process and then explored a variety of special types of error control procedures. While Cushing's analysis seems to be far removed from the work of Yu and Neter, it is possible to recast the Cushing analysis in the matrix notation of Yu and Neter (see section A1.3).

A combined set of "accounting functions" for modeling the flow of documents through an i.c.s. can be developed using a two state version of the Yu and Neter approach, a matrix notation version of the Cushing error control procedures, and a few additional features. Each of these accounting functions is composed of several logically related processing steps. Each processing step is in turn defined by a transitional matrix such as illustrated by (1). The mathematical details of these accounting functions are defined in the next chapter. There are accounting functions for tandem and parallel processing steps of a document, for merging two document flows, for consolidating two documents into a composite document and for an error control analysis of documents.

Using these functions an i.c.s. can be viewed as a directional network, where each node represents a state probability vector and each link represents one of the above document functions. Thus given perfect knowledge of all the probabilities embedded in each function, the auditor can determine the single number that represents the probability that a document (or accounting transaction) will emerge from the network in an error free state.

These functions do not consider an auditor's uncertainty about the nature of the probabilistic process. A given set of functions for a particular i.c.s. with numerically specified error rates is just one realization of the auditor's joint probability judgment for all component probabilities of the processing steps of the i.c.s. Auditing oriented functions for studying i.c.s. should recognize both the probabilistic nature of the process and the auditor's uncertainty about the component probabilities of the process.

The mathematical functions developed in the next chapter provide this capability. Rather than leading to a single number representing the probability that a transaction is an "error," these auditing functions can be used to determine a p.d.f. for this error rate probability.

What one defines as an error is immaterial to the form of the mathematical analysis. However, it is useful to consider a document in error if there is information on the document that without correction will lead to a dollar error in an accounting transaction. For example, an erroneous number of overtime hours on a time card is a dollar generating error. A bridge between this definition of an error and the auditor's more traditional consideration of compliance errors is developed in section 4.3.

4.2.2 Statistical Independence

The analysis of the next chapter leads to two different types of probability distributions each with associated questions of statistical dependence. At the accounting function level of Yu and Neter there could be statistical dependence not captured by the transitional probability matrix. At the auditing function level there are questions pertaining to the statistical dependence of the auditor's uncertainty about the component error rates of the accounting functions.

In a Yu and Neter type analysis it is assumed that there is statistical independence between the probability states of different transitional probability matrices. For example, given that a document is in state s_j^2 at the start of the processing step 3, the assumption implies that

$$P(s_k^3|s_j^2,s_i^1) = P(s_k^3|s_j^2)$$

where the superscripts 1, 2, and 3 indicate three tandem processing steps that result in states s_i^1, s_j^2 and s_k^3. Thus each processing step is assumed to be a Markovian process with a memoryless transitional matrix. The probability elements of this matrix depend upon the input state but are statistically independent of the circumstances that might lead to this input state.

Bodnar (1975) has discussed two aspects of this assumption. Under his interpretation it is assumed that there are no dependent production failures (i.e., incompetence or a faulty system design). Further there is an assumption of no collusion, or "people failures" as he calls this and other fraudulent activities. He cites and reiterates the viewpoint of Meister (1964) and Carmichael (1970) that this assumption definitely narrows the scope of such models.

No attempt is made in this dissertation to model fraudulent collusion. In the opinion of the author, such an analysis would not be directed at the operation of an identifiable process, but rather at modeling the circumstance under which this process is a charade for the actual operation of the system. Questions of fraudulent collusion are concerned with identifying various possible unknown systems, not necessarily with assigning probabilities to each potential system.

The analysis of the next chapter accommodates certain forms of statistical dependence between processing steps of an i.c.s. caused by production failures. This is accomplished by assuming that each (error rate) probability of (1) is a r.v. with a probability distribution, and by defining broad accounting functions that include several tandem processing steps. Procedures are then developed for accommodating the statistical dependence that may arise between the various probabilities within each accounting function. The analysis does not allow for statistical dependence between separate accounting functions.

Within an accounting function the statistical dependence that might exist between two error rate r.vs. may be due to production failures or to the auditor's judgmental uncertainty or possibly due to both of these causes. Thus, even if there is no reason to expect that knowledge of one state influences the probability of another state, knowledge of the probability of the first state may affect the auditor's probability distribution for the probability of the second state. That is, while the actual probabilistic processes may not be correlated, both processes may be subject to the same type of environmental influences. Consequently, the expertise developed by the auditor in one process may be transferable to the second process.

As a result of the types of statistical dependences that are analyzed, the procedures developed in the next chapter can always be used to study isolated auditing functions without compromising the analysis by assuming statistical independence. However, these auditing functions generally cannot be used to study an i.c.s. for which it is imperative to recognize the statistical dependence that might exist between error rate r.vs. drawn from separate auditing functions.

In the early stages of an initial auditing engagement, when the auditor may have high levels of uncertainty about the reliability of an i.c.s., the assumption of independence between the error rates of several auditing functions may be unrealistic. Knowledge of a high error rate in one function could affect the auditor's uncertainty about error rates in other functions. Subsequent to this phase of the audit, exact knowledge of one error rate in one function is not as likely to affect the auditor's now less diffused uncertainty about another function. As more evidence is collected about an error probability one expects that the subsequent knowledge of the remaining uncertainty will have less of a global consequence on the auditor's uncertainty about other error rates.

4.2.3 Monte Carlo Simulation

The mathematical methodology developed in the next chapter is based upon the moment properties of the component p.d.fs. of an i.c.s. Alternatively, one could derive the p.d.f. of a summary error rate r.v. for an i.c.s. through Monte Carlo simulation. With such a stochastic approach each simulated error rate is the result of first sampling all the component p.d.fs. once and then combining these numerical realizations into a composite error rate. With a sufficiently large number of generations of these composite error rates it is possible to approximate the composite p.d.f.

It is not clear, given current computer technology, that Monte Carlo simulation is an economically viable approach when extensive sensitivity analysis is desired. These considerations are particularly applicable in an audit environment, where a mathematical model of an i.c.s. may act as a mechanism for exploring the implications of variations in the auditor's judgment of his uncertainty rather than for finding a single solution.

There are further difficulties with a Monte Carlo approach. Each particular type of component p.d.f. of a Monte Carlo simulation requires its own special considerations as to how it can be best sampled. In addition, unlike a moment approach, one cannot match the precision of the input data to the precision required of the output p.d.f. Regardless of these output requirements one needs a precise specification of the input component p.d.fs. Finally, in conducting a sensitivity analysis, each variation usually requires a completely new simulation.

4.3 The Relationship of Error Rates to Traditional Measures of Compliance

The auditing functions developed in the next chapter are stated in terms of the conditional probabilities of error for individual documents of an i.c.s. These error rates give the probability that a document initially in or not in error will leave a processing step in error.

This formulation is not directly equivalent to the auditor's traditional emphasis on reviewing the i.c.s. and measuring compliance to the prescribed procedures. A bridge between these two viewpoints is developed in this section.

4.3.1 The Internal Control System Perspective

As discussed in S.A.S. No. 1 of the AICPA, the study and evaluation of internal controls encompasses the review of the system of internal controls and tests of compliance to the prescribed methods. In analytically interpreting these standards Kinney (1975) draws upon the analysis of Cushing (1974) and focuses on the design reliability of the system and rate of compliance of the system with this design.

In studying the design reliability the auditor reviews the system to determine the possibility of errors given that there is full knowledge of and compliance with the prescribed methods. In studying the level of compliance the auditor is concerned with the degree to which the "procedures and prescribed methods . . . are in use and are operating as planned" (AICPA S.A.S. No. 1, §320.50).

One of the difficulties with this dichotomized analysis is to determine how these two measures of system performance relate to the accuracy of account balances. Kinney utilized an heuristic functional relationship. Bailey and Jensen (1977) avoid some of these analytical problems by considering a binary compliance variable. While it is meaningful to consider if an individual document is processed in compliance with the prescribed procedures, it is not clear in their analysis what it means to state than an i.c.s. is in or is not in compliance.

These difficulties are avoided in this dissertation by utilizing the individual documents processed by the i.c.s. as the unit of analysis. Thus, the analysis does not consider if an i.c.s. is or is not in compliance, or what the rate of compliance is for the i.c.s. Rather, the analysis considers for each processing step within the i.c.s. the probability that a document is in compliance with the prescribed control procedures, and the probability that it contains a defect that is or will lead to a dollar error. This focus on individual documents is compatible with the analysis of chapter 6 for integrating evidence from several i.c.ss. with direct evidence of errors in account balances.

4.3.2 The Document Level of Analysis

The mathematical formulation in the next chapter is based upon probability distributions for the probability of a document leaving each processing step in error (E). These probabilities (or error rates) are conditional on the in error (D_e) and not in error (D_{ne}) initial state of the document when it entered the processing step. The mathematical formulation is thus based upon probabilities distributions for the probabilities $P(E|D_e)$ and $P(E|D_{ne})$. The subsequent discussion of this paragraph relates these conditional probabilities to the design reliability and compliance emphasis of traditional auditing.

It is convenient to consider the following three binary states pertaining to a document's flow through a processing step of an accounting function. The document can leave the processing step in error (E) or not in error (~E). The processing of the document within the processing step can be in compliance (C), or not in compliance (~C), with the prescribed control procedures. And the tic marks, initials, etc., of the audit trail can assert that there was compliance (AC), or not assert that there was compliance (~AC) with the prescribed procedures. Letting D denote either D_e or D_{ne} the interrelationship between these states is given by

$$P(E|D) = P(E|C,D)P(C|D) + P(E|{\sim}C,D)P({\sim}C|D) \tag{1}$$

$$P(C|D) = P(C|AC,D)P(AC|D) + P(C|{\sim}AC,D)P({\sim}AC|D) \tag{2}$$

The distinction made in these equations between asserted compliance and actual compliance emphasize the nature of the different types of evidence the auditor tends to collect in compliance testing. Thus a sample of tic marks (AC) does not necessarily imply that there was compliance. Or even if compliance is assured, it does imply that all errors have been detected. The probabilities $P(C|D)$ and $P(AC|D)$ are related to the auditor's traditional measures of compliance. The $P(E|C,D)$ is a measure of design reliability.

Three aspects of the above formulation warrant special consideration. First, the emphasis on individual processing steps corresponds to the detailed analysis of specific errors emphasized by Moriarity (1975). Note, however, that during the usual preliminary review of an auditor, where an aggregate i.c.s. perspective is adopted (e.g., the analysis of Kinney 1975), the total i.c.s. can be considered as a single processing step and analyzed using the logic of this dissertation. Then, when problems are encountered, the more detailed focus of the auditing functions can be utilized.

Second, the auditor must specify the probability of a document being in error, rather than specifying an error rate for all documents that are processed. This latter approach is required for example by Kinney's (1975) analysis. In practice the distinction may not be significant and of interest to the auditor. In both cases, the auditor must recognize his uncertainty in the specified value.

Third, conditional probabilities are required in this analysis. Thus it is possible to make a distinction between the clerical results of processing a document that was initially in error or was initially free of error. If this distinction is not desired then the auditor can set $P(C|D_e) = P(C|D_{ne}) = P(C)$. With these assumptions this aspect of the analysis conforms to standard practice.

In a well designed system with positive assurance that asserted compliance is equivalent to actual compliance, it may be possible to assume that

$$P(E|C,D_{ne}) = 0 \qquad P(C|AC,D) = 1 \qquad P(C\sim|AC,D) = 0 \tag{3}$$

Consequently the probability of error given by (1) is just the product of the probability of a lack of compliance and the probability of an error given a lack of compliance when $D = D_{ne}$.

When both of these probabilities are viewed as unknown parameters of Bernoulli processes, a natural conjugate analysis for each probability leads to two separate beta distributions (Raiffa and Schlaifer 1961, p. 263). Exact and approximating distributions for the resulting product of beta r.vs. have been studied by numerous authors under the assumption that the component beta distributions are statistically independent. This literature is reviewed briefly in paragraph A2.5.4.

When a very tractable expression for the product is desired it is suggested in this literature that the product be approximated by a single beta distribution with the same first two moments. However, given the assumption of statistical independence, the moments of a beta product are just the product of the beta moments. Consequently, the exact moments for (1) and (2) rather than those of an approximation can be used as inputs to the auditing function.

As the assumptions of (3) are progressively relaxed, the informational demands on the auditor become more cumbersome. When each probability of (1) is a r.v. with known moments, the moments of the r.v. $P(E|D)$ can be found through repeated use of the binomial theorem. Dropping the explicit display of the prior state, this yields

$$E[P(E)^n] = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} (-1)^{k_2} \binom{n}{k_1} \binom{n-k_1}{k_2} E\left[P(C)^{k_1+k_2} P(E|C)^{k_1} P(E|\sim C)^{n-k_1}\right] \tag{4}$$

When $P(C)$ can be further decomposed using (2) it follows that

$$P(C)^{k_1+k_2} = \sum_{k_3=0}^{k_1+k_2} \sum_{k_4=0}^{k_1+k_2-k_3} (-1)^{k_4} \binom{k_1+k_2}{k_3} \binom{k_1+k_2-k_3}{k_4}$$

$$\cdot\ P(A)^{k_3+k_4}\ P(C|A)^{k_3}\ P(C|\sim A)^{k_1+k_2-k_3} \tag{5}$$

When statistical independence exists between r.vs. equations (4) and (5) lead to a weighted sum of the product of the component r.v. moments. Evaluation procedures that can be used under conditions of statistical dependence are discussed in section A1.4.

The chain of conditional probabilities of (1) and (2) and the informational demand of (4) and (5) emphasize the relative weakness of assertive compliance (AC) evidence. In applying these procedures many of the intermediate conditional probability distributions may have to be specified solely using judgmental methods. The occurrence of the required conditions of the conditional probabilities may be too infrequent and difficult to locate to allow a statistical analysis to be conducted.

In general the p.d.fs. that are associated with each auditing function can arise from any combination of prior or posterior audit judgments and statistical test data. A general discussion of some of the issues involved in assessing such distributions is given by Winkler (1972, pp. 182-189). Felix (1976) provides a more comprehensive summary and gives some experimental results that focus on the Bernoulli processes and the beta p.d.f. Review articles by Chesley (1975) and Hogarth (1975) were previously cited in section 1.2.

4.4 Analytical Issues in the Determination of Error Rate Probability Distributions

In addition to the judgmental specification issues noted briefly in the last section, there are a number of mathematical considerations that arise in using the beta p.d.f. and other types of p.d.fs. to express the auditor's uncertainty about error rates, conditional probabilities, etc. This section brings together a number of analytical details discussed in greater depth in appendix 2. The discussion considers first the standardized beta p.d.f. and then focuses on other types of p.d.fs.

4.4.1 Use of the Beta Probability Density Function

In most applications an auditor should be able to approximate his prior judgment quite adequately with a beta distribution. Weiler (1965) has demonstrated this robustness in a number of test cases. With moderate priors (i.e., neither highly leptokurtic or extremely diffuse), and samples of 100 observations his plots of prior and posterior density functions show clearly the insensitivity of the Bayesian revision to the prior p.d.f. Weiler also developed several nomographs that are useful in establishing prior beta distributions.

Appendix 2 presents a number of properties of beta distributions and moments that can be used in specifying and calculating the uncertainty associated with each process step. In the simplest case a processing operation is analyzed using a prior to posterior analysis based upon a beta p.d.f. and binomial sample evidence (e.g., Winkler 1972, pp. 149-159).

Empirical error rate frequency data can be fitted to a beta p.d.f. using equations A2.3(12) and A2.3(13) with a = 0 and b = 1. These equations give the parameters of the beta p.d.f. in terms of the distribution's mean and variance which can be estimated from the frequency data. If the auditor's error rate uncertainty can be specified in terms of the mode and variance the procedure of paragraph A2.5.1 can be used to fit a beta p.d.f. This analysis is a variation of the PERT approach where a beta distribution is specified using the distribution's mode and a percentile range (Malcolm, Roseboom and Clark 1959; Moder and Rodgers 1968).

A relative measure of the value of alternative prior beta p.d.fs. for a potential sample size is given by the information ratio discussed in paragraph A2.5.2. This procedure gives for two prior beta distributions the ratio of the expected value of the posterior variances for a given sample size. Using this ratio one can determine the degree to which two possible priors are expected to lead to similar posterior p.d.fs. Thus, if the information ratio indicates that the expected after sampling results are very similar there is little reason to linger over the prior assessment.

In studying the significance of an auditor's prior or posterior beta p.d.f. it is often useful to consider the probability mass in the upper tail of the distribution corresponding to higher error rates. Equations A2.5(11) and A2.5(13) give two possible approximations to this probability. The second equation based upon an approximation by Boyd (1971) is very simple to use and appears to be vastly superior to the direct calculation of the probability. The direct approach based upon the definition of a beta p.d.f. often leads to numerical problems with the highly skewed, leptokurtic beta distributions of interest in auditing research.

4.4.2 Other Probability Density Functions

When an improper or nonnatural conjugate prior p.d.f. is used with statistical test data of error rates the moment method discussed in paragraph A2.5.5 can be used to apply Bayes' law. Or alternatively a numerical approximation based on a discrete approximation of the prior p.d.f. can be used (see for example, Schlaifer 1961, pp. 194-195; Winkler 1972, pp. 192-197). The moment method is ideally suited for prior p.d.fs. approximated by a lognormal distribution. In other cases it must be numerically feasible to calculate the higher moments of the prior distribution. Using A2.2(5) and the recursive formula A2.2(1) these higher moments can be calculated for a prior p.d.f. approximated by an extended beta distribution.

Such a prior distribution could be useful when it is desirable to increase the precision of a prior specification of judgment from that possible with a beta distribution defined on the interval [0,1]. By letting the upper limit vary below the standardized value of 1 the robustness of the beta distribution can be significantly increased. The procedure given in paragraph A2.3.2 can be used with $a = 0$ to fit the extended beta distribution to mean, variance and skewness effects represented by the first three noncentral moments.

4.5 Concluding Remarks

Perhaps more important than the detailed description of a mathematical model is a discussion of the setting in which it can be used. The implications of the assumptions made, the possible alternative approaches, and how the inputs can be developed are major technical aspects of this setting. These are, of course, difficult considerations that can never by completely evaluated. In this chapter these technical issues have been briefly explored. This preliminary evaluation should make the subsequent mathematical analysis of the next chapter more meaningful.

This subsequent analysis might be used to better understand an accounting process with or without narrowing the scope of the auditor's uncertainty. Or, it is possible that these techniques could act as an aid in making auditing decisions more efficiently. Finally, there is always the possibility that these procedures could even improve the quality of the auditor's attestment decision. In chapter 8, as part of the concluding discussion, several broader issues pertaining to the normative or descriptive aspects of these techniques are discussed.

CHAPTER 5

A MATHEMATICAL ANALYSIS OF AN AUDITING MODEL OF AN INTERNAL CONTROL SYSTEM

5.1 Preface

The analysis in this chapter starts with a description of a set of accounting functions that can be used to model the processing steps of an i.c.s. These accounting functions are partially drawn from the analysis of Yu and Neter, and Cushing.

It is assumed in the third section that these functions are also auditing functions. Thus each error rate within each of these functions is a r.v. with a p.d.f. specifying the auditor's uncertainty about the error rate. The probability moments of these individual error rate distributions can be used to determine the summary moments of each auditing function. In the third section a set of moment functions are derived for finding these probability moments. Several special case simplifications of these moment functions are also presented.

A procedure is discussed in the fourth section for combining the summary moments from a series of these auditing functions. This leads to aggregate moments for the accumulative error rate of a series of processing steps. When these processing steps completely define an i.c.s., these aggregate moments define a composite error rate p.d.f. for the probability of errors in the journal entries emerging from the complete i.c.s.

An orthogonal polynomial procedure developed in appendix 3 can then be used to asymptotically determine the form of the composite error rate p.d.f. from the aggregate moments. The particular form of this asymptotic series expansion is analytically compatible with the subsequent analysis of account balances presented in chapter 6.

5.2 Accounting Functions for an Internal Control System

Using the two state version of the Yu and Neter notation discussed in paragraph 4.2.1, state vectors are represented by (P,Q) and $(P,Q)_N$ the (not in error, in error) vector before and after a processing step. The transitional matrix is represented by

$$\begin{array}{lcc} & \multicolumn{2}{c}{\text{After Processing}} \\ & \text{NE} & \text{E} \\ \text{Before Processing} \begin{array}{c} \text{NE} \\ \text{E} \end{array} & \multicolumn{2}{c}{\begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix}} \end{array}$$

NE = Not in Error
E = In Error

where the marginal notation specifies the row and column conventions being used. For the simplest type of accounting function composed of one normal processing step it follows that

$$(P,Q)_N = (P,Q) \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} = (P p_1 + Q p_2,\ P q_1 + Q q_2)$$

In particular the error rate of the normal processing step is given by

$$Q_N = (1-Q)\, q_1 + Q q_2 \tag{1}$$

where P = 1 - Q.

Table 5.2.1 presents seven accounting functions that can be used to model the document flow of an i.c.s. These will be subsequently discussed after defining the notation and format of this table. Capital P and Q are used to indicate the probabilities of the no error and in error states prior and subsequent to a particular accounting function. Lower case p and q are used to indicate the component error rate parameters for each process. For all subscripts i, $P_i + Q_i = 1$ and $p_i + q_i = 1$. The first line of each function represents symbolically how the initial state vector is affected by the accounting function. The second line defines mathematically this effect.

The OR rejoining notation of functions (d) and (e) indicate that a single document is being formed out of two separate documents. Consequently the new document is in error if there is an error in either of the component documents. Alternatively this process can be considered as the formation of a not in error document according to AND logic. Both documents must be correct for the composite document to be correct.

Accounting functions (a), (c), (d) are directly equivalent in a two state model to the functions given by Yu and Neter. They describe the modification to the error probability vector caused by normal processing of documents, the accumulation of documents from two sources and the consolidation of two documents into a new composite document.

Functions (b), (e) and (f) are embellishments to the Yu and Neter system that are useful in an auditing environment. Function (b) expresses two processing steps as a single accounting function. This function can be used to accommodate the statistical dependence that might exist between the component r.vs. of two normal processing functions (i.e., type a) operating in tandem. Function (e) accommodates the

Table 5.2.1

Accounting Functions for Modeling Internal Control Systems

(a) Normal Processing of a Document

$$(P,Q) \rightarrow \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} \rightarrow (P,Q)_N$$

$$(P,Q)_N = (P,Q) \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} = (Pp_1+Qp_2,\ Pq_1+Qq_2)$$

(b) Tandem Processing of a Document

$$(P,Q) \rightarrow \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} \rightarrow \begin{pmatrix} p_3 & q_3 \\ p_4 & q_4 \end{pmatrix} \rightarrow (P,Q)_T \sim (P,Q) \begin{pmatrix} p & q \\ \bar{p} & \bar{q} \end{pmatrix}_T = (P,Q)_T$$

$$\begin{pmatrix} p & q \\ \bar{p} & \bar{q} \end{pmatrix}_T = \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} \begin{pmatrix} p_3 & q_3 \\ p_4 & q_4 \end{pmatrix} = \begin{pmatrix} p_1p_3+q_1p_4 & p_1q_3+q_1q_4 \\ p_2p_3+q_2p_4 & p_2q_3+q_2q_4 \end{pmatrix}$$

(c) Weighted Average Merging of Document Flows

$$\begin{tikzcd} (P_1,Q_1) \arrow[dr, "w_1"] & & \\ & {} \arrow[r] & (P,Q)_W \\ (P_2,Q_2) \arrow[ur, "w_2"'] & & \end{tikzcd} \qquad w_1 + w_2 = 1$$

$$(P,Q)_W = w_1(1-Q_1,Q_1) + w_2(1-Q_2,Q_2) = [1-(w_1Q_1+w_2Q_2),\ w_1Q_1 + w_2Q_2]$$

(d) OR Rejoining of Two Documents

$$\begin{tikzcd} (P_1,Q_1) \arrow[dr] & & \\ & {} \arrow[r, "\text{OR Error}", "\text{Logic}"'] & (P,Q)_R \\ (P_2,Q_2) \arrow[ur] & & \end{tikzcd}$$

$$(P,Q)_R = [(1-Q_1)(1-Q_2),\ 1 - (1-Q_1)(1-Q_2)] = [(1-Q_1)(1-Q_2),\ Q_1 + Q_2 - Q_1Q_2]$$

Table 5.2.1 (continued)

(e) Parallel Processing with OR Rejoining of Errors

$$(P,Q) \begin{array}{l} \nearrow \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} \searrow \\ \searrow \begin{pmatrix} p_3 & q_3 \\ p_4 & q_4 \end{pmatrix} \nearrow \end{array} \xrightarrow[\text{Logic}]{\text{OR Error}} (P,Q)_P$$

Letting $\cap$ symbolize OR logic for the error states this processing step can be written

$$(P,Q)_P = (P,Q) \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} \cap (P,Q) \begin{pmatrix} p_3 & q_3 \\ p_4 & q_4 \end{pmatrix}$$

$$= (Pp_1+Qp_2, Pq_1+Qq_2) \cap (Pp_3+Qp_4, Pq_3+Qq_4)$$

$$Q_P = [(Pq_1+Qq_2) + (Pq_3+Qq_4)] - (Pq_1+Qq_2)(Pq_3+Qq_4)$$

$$P_P = (Pp_1+Qp_2)(Pp_3+Qp_4)$$

(f) Alternative Processing of a Document

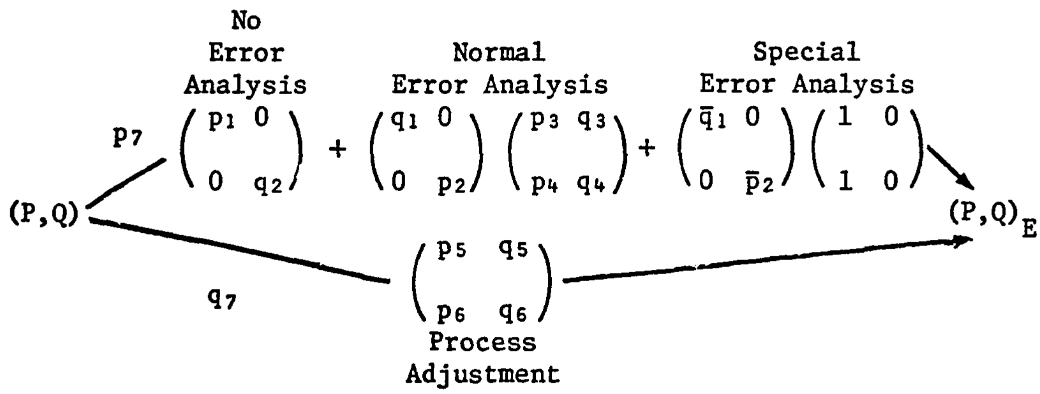
$$(P,Q) \begin{array}{l} \nearrow \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} \searrow^{w_1} \\ \searrow \begin{pmatrix} p_3 & q_3 \\ p_4 & q_4 \end{pmatrix} \nearrow_{w_2} \end{array} \longrightarrow (P,Q)_A \qquad w_1 + w_2 = 1$$

Table 5.2.1 (continued)

$$(P,Q)_A = (P,Q)\left[w_1\begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} + w_2\begin{pmatrix} p_3 & q_3 \\ p_4 & q_4 \end{pmatrix}\right]$$

$$= [P(w_1p_1+w_2p_3) + Q(w_1p_2+w_2p_4),\ P(w_1q_1+w_2q_3) + Q(w_1q_2+w_2q_4)]$$

(g) Error Control Processing



$$(P,Q)_E = (P,Q)\left[p_7\begin{pmatrix} p_1+q_1p_3+\bar{q}_1 & q_1q_3 \\ p_2p_4+\bar{p}_2 & q_2+p_2q_4 \end{pmatrix} + q_7\begin{pmatrix} p_5 & q_5 \\ p_6 & q_6 \end{pmatrix}\right]$$

$$= [P(p_7(p_1+q_1p_3+\bar{q}_1)+q_7p_5) + Q(p_7(p_2p_4+\bar{p}_2)+q_7p_6),$$

$$P(p_7q_1q_3+q_7q_5) + Q(p_7(q_2+p_2q_4)+q_7q_6)]$$

statistical dependence created when duplicate copies of a document are subject to separate processing and then rejoined. Function (f) can be used when certain documents are given special handling. For large dollar amounts this function might be applied with $q_3 = q_4 = 0$.

Function (g) is an extension of the Yu and Neter error control processing for more elaborate types of processing. In this function the probability p_7 can be used to explicitly recognize less than 100% compliance to an error control step. The possibility that a lack of compliance may influence the accuracy of previous processing step(s) can be accommodated using the process adjustment matrix of (g). Scenarios can easily be developed where the surreptitious deletion of a control step increases or decreases the care exercised in implementing previous processing steps. For example, $q_5 = .03$, $q_6 = 1$ implies that the error rate for correct documents should be reduced by 3%.

With $q_7 = q_1 = p_2 = 0$ (g) reduces to the Yu and Neter form of error control processing. With $q_7 = \bar{q}_1 = \bar{p}_2 = 0$, (g) represents the basic error control configuration of Cushing. The normal and special error control feature of (g) accommodates special documents that are given extra care in error checking.

Functions (a) through (e) include all aspects of the Yu and Neter system for describing an i.c.s. There are, however, several refinements of the Cushing system that cannot be represented by (g). In particular, there is the possibility of expanding (g) or of creating another function that focuses on the feedback error controls of the Cushing system. As represented in (g) all documents are sent forward to the next processing step after error processing.

5.3 Auditing Functions and Their Moments

When each of the error rates of the accounting functions of table 5.2.1 is assumed to be an uncertain value of concern to the auditor, these same accounting functions can be considered as auditing functions composed of algebraic functions of r.vs. Now when several such functions are linked together to describe the flow of documents through an i.c.s. an extremely complex problem arises in determining the probability distribution of the summary error rate r.v. The approach taken in this dissertation to this problem is to determine the probability moments for each of the auditing functions of table 5.2.1.* These moments are then consolidated into summary moments for the error rate of a complete i.c.s.

Using the binomial theorem, the probability moments for an error rate r.v. emerging from an auditing function can be expressed in terms of the moments of the component r.vs. of the function and the moments of the initial error rate. For example, under auditing conditions equation 5.2(1) becomes

$$\tilde{Q}_N = (1-\tilde{Q})\tilde{q}_1 + \tilde{Q}\tilde{q}_2$$

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*The general problem of collapsing into a summary distribution a complex algebraic function of continuous r.vs. can be approached in several alternative ways. Heck and Tung (1962) discussed an empirical approach that sequentially combines discrete interval approximations for the known component p.d.fs. Abraham and Prasad (1969) and Prasad (1970) have developed a combined Mellen/Laplace transformation procedure. Neither of these approaches appear to have led to any subsequent published comments. They do not appear easy to implement or computationally efficient. Neither gives results that can be immediately translated into a p.d.f. for the summary r.v. Another approach due to King, Sampson and Simms (1975) is very tractable, but it is based upon a very restrictive p.d.f. model of the component r.vs.

where the ~ signs have been added to emphasize the r.v. nature of the identity. By repeated use of the binomial theorem the moments of Q_N can be calculated. Thus:

$$E(Q_N^n) = E\left((1-Q)q_1 + Qq_2\right)^n = \sum_{i=0}^{n} \binom{n}{i} E\left((1-Q)^{n-i} q_1^{n-i} Q^i q_2^i\right)$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-i} (-1)^{n-i-j} \binom{n}{i} \binom{n-i}{j} Y \tag{1}$$

where all expectations are with respect to the joint p.d.f. of (Q,q_1,q_2), and as a result of the assumption of statistical independence discussed in paragraph 4.2.2

$$Y = E_{Q,q_1,q_2}(Q^{n-j} q_1^{n-i} q_2^i) = E_Q(Q^{n-j}) E_{q_1,q_2}(q_1^{n-i} q_2^i) \tag{2}$$

If the r.vs. $\tilde{q}_1$ and $\tilde{q}_2$ are statistically independent then $E_{q_1 q_2} = E_{q_1} E_{q_2}$ and (2) can be easily calculated. Otherwise the procedures suggested in A1.4 could be used to approximate the form of dependence that may exist between r.vs. $\tilde{q}_1$ and $\tilde{q}_2$.

Table 5.3.1 develops expressions for the probability moments of each output error rate r.v. of table 5.2.1. While somewhat tedious the analysis completely parallels the discussion and conclusions represented by (1) and (2). A more condensed expected value notation has been used without the r.v. subscripts used in (2).

The assumptions of statistical independence discussed in paragraph 4.2.2 are assumed throughout this analysis. If it is further assumed that the statistical processes that generate the component error rates of each auditing function are statistically independent, then the moments of the component error rate r.vs. can be used to evaluate the joint moments of equations (3) through (11) of table 5.3.1. As previously noted, section A1.4 discusses possible models for incorporating statistical correlation between component r.vs. when independence cannot be assumed.

Function (e) for parallel processing with OR logic rejoining can lead to some computational difficulties. From (8) of table 5.3.1 it follows that in order to determine $E(Q_p^n)$ it is necessary to know $E(Q^i)$ for $i = 1,\ldots, 2n$. Thus, at all previous processing steps twice as many moments must be calculated as is intended to be used in the final moment approximation for the p.d.f. of the aggregate error rate. One possible approximation that avoids this difficulty is to fit a four parameter beta distribution to the input error rate r.v. using $E(Q^i)$ $i = 1,2,3,4$, as discussed in paragraph A2.3.4. Assuming that the input random variable is unimodal, the higher moments of the fitted distribution given in paragraph A2.2.2 may be a tolerable approximation for the unknown moments.

While the notation of these moment functions is tedious, the empirical calculation of moments is straightforward using nested FORTRAN 'DO LOOPS.' The binomial coefficients for $n \leq 20$ given in table 5.3.2 can be input to a computer program as a two dimension array. As seen from table 5.3.3 functions (b), (e), (f) and (g) generate a large number of terms. However, the equations can be considerably simplified when it can be assumed that some of the component error rates do not significantly differ from zero.

Table 5.3.1

Moment Functions for Internal Control Systems

(a) Normal Processing of a Document

$$E(Q_N^n) = E((Pq_1+Qq_2)^n) = E(\sum_{i=0}^{n} \binom{n}{i} (1-Q)^{n-i} q_1^{n-i} Q^i q_2^i)$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-i} (-1)^j \binom{n}{i} \binom{n-i}{j} E(Q^{i+j}) E(q_1^{n-i} q_2^i) \tag{3}$$

(b) Tandem Processing of a Document

$$E(Q_T^n) = E[((1-Q)(p_1q_3 + q_1q_4) + Q(p_2q_3 + q_2q_4))^n]$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-i} (-1)^j \binom{n}{i} \binom{n-i}{j} E[Q^{i+j} (p_1q_3 + q_1q_4)^{n-i} (p_2q_3+q_2q_4)^i] \tag{4}$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i} \sum_{\ell=0}^{n-i-k} \sum_{m=0}^{i} \sum_{r=0}^{i-m} (-1)^{j+\ell+r} \tag{5}$$

$$\cdot \binom{n}{i} \binom{n-i}{j} \binom{n-i}{k} \binom{n-i-k}{\ell} \binom{i}{m} \binom{i-m}{r} E(Q^{i+j}) E(q_1^{\ell+k} q_2^{r+m} q_3^{n-k-m} q_4^{k+m})$$

Note that the derivation of (4) and (5) is just a repetition of (3) with changes of notation.

(c) Weighted Average Merging of Document Flows

$$E(Q_W^n) = E((w_1Q_1+w_2Q_2)^n) = \sum_{i=0}^{n} \binom{n}{i} w_1^{n-i} w_2^i E(Q_1^{n-i} Q_2^i) \tag{6}$$

(d) OR Rejoining of Two Documents

$$E(Q_R^n) = E((Q_1+Q_2-Q_1Q_2)^n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} E((Q_1+Q_2)^{n-i} Q_1^i Q_2^i)$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-i} (-1)^i \binom{n}{i} \binom{n-i}{j} E(Q_1^{n-j} Q_2^{i+j}) \tag{7}$$

Table 5.3.1 (continued)

(e) Parallel Processing with OR Rejoining

$E(Q_P^n)$

$$= E[((Pq_1+Qq_2)+(Pq_3+Qq_4)-(Pq_1+Qq_2)(Pq_3+Qq_4))^n]$$

$$= \sum_{i=0}^{n} (-1)^i \binom{n}{i} E[((Pq_1+Qq_2)+(Pq_3+Qq_4))^{n-i}(Pq_1+Qq_2)^i(Pq_3+Qq_4)^i]$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-i} (-1)^i \binom{n}{i}\binom{n-i}{j} E[(Pq_1+Qq_2)^{n-j}(Pq_3+Qq_4)^{i+j}]$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{k=0}^{n-j} \sum_{\ell=0}^{i+j} (-1)^i \binom{n}{i}\binom{n-i}{j}\binom{n-j}{k}\binom{i+j}{\ell} E(Y)$$

where $E(Y) = E[(1-Q)^{n+i-k-\ell}Q^{k+\ell}]E[q_1^{n-j-k}q_2^k q_3^{i+j-\ell}q_4^{\ell}]$

Now $E[(1-Q)^{n+i-k-\ell}Q^{k+\ell}] = \sum_{m=0}^{n+i-k-\ell} (-1)^m \binom{n+i-k-\ell}{m} E(Q^{k+\ell+m})$

Thus $E(Q_P^n)$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{k=0}^{n-j} \sum_{\ell=0}^{i+j} \sum_{m=0}^{n+i-k-\ell} (-1)^{i+m} \binom{n}{i}\binom{n-i}{j}\binom{n-j}{k}\binom{i+j}{\ell}\binom{n+i-k-\ell}{m}$$

$$\cdot\ E(Q^{k+\ell+m})E(q_1^{n-j-k}q_2^k q_3^{i+j-\ell}q_4^{\ell}) \qquad (8)$$

(f) Alternative Processing of a Document

$E(Q_A^n)$

$$= E[(P(w_1q_1+w_2q_3) + Q(w_1q_2+w_2q_4))^n]$$

Table 5.3.1 (continued)

$$= \sum_{i=0}^{n} \binom{n}{i} E[(1-Q)^{n-i}(w_1q_1+w_2q_3)^{n-i}Q^i(w_1q_2+w_2q_4)^i]$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i} \sum_{\ell=0}^{i} (-1)^j \binom{n}{i}\binom{n-i}{j}\binom{n-i}{k}\binom{i}{\ell} w_1^{n-k-\ell} w_2^{k+\ell}$$

$$\cdot\ E(Q^{i+j})E(q_1^{n-i-k} q_2^{i-\ell} q_3^k q_4^\ell) \tag{9}$$

(g) Error Control Processing

$E(Q_E^n)$

$$= E[(P(p_7q_1q_3+q_7q_5)+Q(p_7(q_2+p_2q_4)+q_7q_6))^n]$$

$$= \sum_{i=0}^{n} \binom{n}{i}(1-Q)^{n-i}(p_7q_1q_3+q_7q_5)^{n-i}Q^i(p_7(q_2+p_2q_4)+q_7q_6)^i$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i} \sum_{\ell=0}^{i} (-1)^j \binom{n}{i}\binom{n-i}{j}\binom{n-i}{k}\binom{i}{\ell}$$

$$\cdot\ E[Q^{i+j}]E[(1-q_7)^{n-k-\ell} q_7^{\ell+k}(q_1q_3)^{n-i-k} q_5^k (q_2+p_2q_4)^{i-\ell} q_6^\ell]$$

Now $(1-q_7)^{n-k-\ell} = \sum_{m=0}^{n-k-\ell} (-1)^m \binom{n-k-\ell}{m} q_7^m$ and

$$(q_2+p_2q_4)^{i-\ell} = \sum_{r=0}^{i-\ell} \binom{i-\ell}{r} q_2^{i-\ell-r} p_2^r q_4^r$$

Thus $E(Q_E^n)$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i} \sum_{\ell=0}^{i} \sum_{m=0}^{n-k-\ell} \sum_{r=0}^{i-\ell} (-1)^{j+m} \binom{n}{i}\binom{n-i}{j}\binom{n-i}{k}\binom{i}{\ell}\binom{n-k-\ell}{m}\binom{i-\ell}{r}$$

Table 5.3.1 (continued)

$$\cdot\ E(Q^{i+j})E(q_7^{\ell+k+m} q_1^{n-i-k} q_2^{i-\ell-r} q_3^{n-i-k} q_4^{r} q_5^{k} q_6^{\ell} p_2^{r}) \tag{10}$$

When $\bar{p}_2 = 0$ and hence $p_2 = 1 - q_2$, equation (8) expands to

$$E(Q_E^n)$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i} \sum_{\ell=0}^{i} \sum_{m=0}^{n-k-\ell} \sum_{r=0}^{i-\ell} \sum_{s=0}^{r} (-1)^{j+m+s} \binom{n}{i}\binom{n-i}{j}\binom{n-i}{k}\binom{i}{\ell}\binom{n-k-\ell}{m}$$

$$\binom{i-\ell}{r}\binom{r}{s} \cdot E(Q^{i+j})E(q_7^{\ell+k+m} q_1^{n-i-k} q_2^{i-\ell-r+s} q_3^{n-i-k} q_4^{r} q_5^{k} q_6^{\ell}) \tag{11}$$

Table 5.3.2

Binomial Coefficients, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for $n \leq 20$

k \ n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2			1	3	6	10	15	21	28	36	45	55	66	78	91	105	120	136	153	171	190
3				1	4	10	20	35	56	84	120	165	220	286	364	455	560	680	816	969	1140
4					1	5	15	35	70	126	210	330	495	715	1001	1365	1820	2380	3060	3876	4845
5						1	6	21	56	126	252	462	792	1287	2002	3003	4368	6189	8568	11628	15504
6							1	7	28	84	210	462	924	1716	3003	5005	8008	12376	18564	27132	38760
7								1	8	36	120	330	792	1716	3432	6435	11440	19448	31824	50388	77520
8									1	9	45	165	495	1287	3003	6435	12870	24310	43758	75582	125970
9										1	10	55	220	715	2002	5005	11440	24310	48620	92387	167960
10											1	11	66	286	1001	3003	8008	19448	43758	92378	184756
11												1	12	78	364	1365	4368	12376	31824	75582	167960
12													1	13	91	455	1820	6188	18564	50388	125970
13														1	14	105	560	2380	8568	27132	77520
14															1	15	120	680	3060	11628	38760
15																1	16	136	816	3876	15504
16																	1	17	153	969	4854
17																		1	18	171	1140
18																			1	19	190
19																				1	20
20																					1

Table 5.3.3

Number of Terms for Each Moment Function

Order of Moment for Function	A,D	B	C	E	F	$G_{(10)}$	$G_{(11)}$
1	3	9	2	14	6	11	13
2	6	42	3	77	20	58	79
3	10	140	4	273	50	210	324
4	15	378	5	748	105	602	1038
5	21	882	6	1729	196	1470	2802
6	28	1848	7	3542	336	3192	6666
7	36	3564	8	6630	540	6336	14388
8	45	6435	9	11571	825	11715	28743
9	55	11011	10	19096	1210	20449	53911
10	66	18018	11	30107	1716	34034	95953

5.4 Simplifications for the Moment Functions

Table 5.4.1 presents optional rules for simplifying the moment functions of table 5.3.1 when particular error rates are zero with probability one. These rules may be particularly useful in an extensive sensitivity analysis. When computation inefficiencies can be tolerated the formulas given in table 5.3.1 can be used without these simplifications provided the convention $0^0 = 0$ is adopted. The results given in table 5.4.1 follow from the derivations for the moment functions given in table 5.3.1. The use of the table is best explained with an example.

For error control processing function (g) assume that $q_3 = q_5 = 0$. The assumption $q_3 = 0$ implies that there is no possibility that a "not in error" document will be incorrectly processed by the error control step. The assumption $q_5 = 0$ implies that a lack of compliance will not increase the probability of an error. Thus, the bypassing of a subsequent error control step does not adversely affect the processing of documents.

From (g) of table 5.4.1 it is seen that $q_3 = 0$ implies that $k = n - i$ and that $q_5 = 0$ implies that $k = 0$. It follows that $k = n - i = 0$ or $i = n$. Thus, $q_3 = q_5 = 0$ implies $i = n$, $k = 0$. Note that the joint entry for $q_3 = q_5 = 0$ also indicates that $j = 0$. With $i = n$ the range of summation of j collapses to $j = 0$. In general if a joint entry exists in the table it should be used, since there are several cases where the joint entry cannot be derived from the component entries. If there is not a joint entry then the component entries completely specify the possible simplifications.

Table 5.4.1

Special Case Rules for Moment Functions

(a) Normal Processing of a Document

For	Set Index	For	Set Index	For	Set Index
$q_1=0$	$i=n,\ j=0$	$q_2=0$	$i=0$	$Q=0$	$i=j=0$

(b) Tandem Processing of a Document

For	Set Index	For	Set Index	For	Set Index
$q_1=0$	$k=0,\ \ell=0$	$q_2=0$	$m=i,\ r=0$	$q_1=q_3=0$	$i=m=n$ $j=k=\ell=r=0$
$q_3=0$	$k=n=i,\ \ell=0$ $m=i,\ r=0$	$q_4=0$	$k=0,\ m=0$	$q_2=q_4=0$	$k=m=r=0$
				$Q=0$	$i=j=m=r=0$

(c) Weighted Average Merging of Document Flows

For	Set Index	For	Set Index
$Q_1=0$	$i=n$	$Q_2=0$	$i=0$

(d) OR Rejoining of Two Documents

For	Set Index	For	Set Index
$Q_1=0$	$i=0,\ j=n$	$Q_2=0$	$i=j=0$

(e) Parallel Processing with OR Rejoining

For	Set Index	For	Set Index	For	Set Index
$q_1=0$	$k=n-j$	$q_2=0$	$k=0$	$q_1=q_2=0$	$i=k=0,\ j=n$
$q_3=0$	$\ell=i+j$	$q_4=0$	$\ell=0$	$q_3=q_4=0$	$i=j=\ell=0$
				$Q=0$	$k=\ell=m=0$

(f) Alternative Processing of a Document

For	Set Index	For	Set Index	For	Set Index
$q_1=0$	$k=n-i$	$q_3=0$	$k=0$	$q_1=q_3=0$	$i=n,\ j=k=0$
$q_2=0$	$\ell=i$	$q_4=0$	$\ell=0$	$q_2=q_4=0$	$i=\ell=0$
				$Q=0$	$i=j=\ell=0$

(g) Error Control Processing

For	Set Index	For	Set Index	For	Set Index
$q_7=0$	$k=\ell=m=0$	$q_1=0$	$k=n-i$	$q_2=0$	$r=i-\ell$
$q_3=0$	$k=n-i$	$q_4=0$	$r=0$	$q_5=0$	$k=0$
$q_6=0$	$\ell=0$	$p_2=0$	$r=s=0$	$Q=0$	$i=j=\ell=r=0$
$q_2=p_2=0$	$r=s=0,\ \ell=i$	$q_2=q_4=0$	$r=s=0,\ \ell=i$		
$q_2=p_2=q_6=0$	$i=\ell=r=s=0$	$q_2=p_2=q_7=0$	$i=k=\ell=m=r=s=0$		
$q_2=q_4=q_6=0$	$i=\ell=r=0$	$q_2=q_4=q_7=0$	$i=k=\ell=m=r=0$		
$q_1=q_5=0$	$j=k=0,i=n$	$q_1=q_7=0$	$k=\ell=m=0,i=n$		
$q_3=q_5=0$	$j=k=0,i=n$	$q_3=q_7=0$	$k=\ell=m=0,i=n$		

Continuing with the example, for fixed index values of $i = n$ and $j = k = 0$ equation (11) of table 5.3.1 reduces to:

$$E(Q_E^n) = \sum_{\ell=0}^{n} \sum_{m=0}^{n-\ell} \sum_{r=0}^{n-\ell} \sum_{s=0}^{r} (-1)^{m+s} \binom{n}{\ell} \binom{n-\ell}{m} \binom{n-\ell}{r} \binom{r}{s}$$

$$\cdot\; E(Q^n) E(q_7^{\ell+m} q_2^{n-\ell-r+s} q_4^r q_6^\ell)$$

Another type of simplification is possible when a series of tandem processing steps conforms to the series configuration of reliability theory. For example, if in table 5.2.1 for (b) it happens that $q_2 = q_4 = 1$ it is easily shown that

$$(P,Q)_T = (P,Q) \begin{pmatrix} p,q \\ \bar{p},\bar{q} \end{pmatrix}_T = (P,Q) \begin{pmatrix} (1-q_1)(1-q_3) & 1-(1-q_1)(1-q_3) \\ 0 & 1 \end{pmatrix} \qquad (13)$$

Setting these parameters to one implies that it is not possible for any existing errors to be corrected during the tandem processing steps.

Now when q_1 and q_3 are independent standardized beta distributions, $(1-q_1)$ and $(1-q_3)$ are also beta distributed with the beta parameters p and q switched. Thus $(1-q_1)(1-q_3)$ is a product of independent beta distributions, a configuration that has been extensively studied. Paragraph A2.5.4 briefly reviews this literature and the conclusion of several authors that a product of beta distributions can be reasonably approximated by a standardized beta distribution with the same first two moments as the product. Since the r.vs. are assumed to be independent these two moments are just the product of the component moments each of which can be calculated using A2.2(1).

The parameters of the approximating beta p.d.f. are then determined using the technique discussed in paragraph A2.3.3. Switching the parameters of the approximation now yields a beta distribution for $1 - (1-q_1)(1-q_3)$. Thus rather than evaluating 5.3(5) of table 5.3.1 with $q_2 = q_4 = 1$ the corresponding moments can be determined using (13). The required moments of the approximating beta p.d.f. are found using equation A2.2(1).

The procedure generalizes in an identical manner to k tandem processing steps in series with $q_2 = q_4 = \ldots = q_{2k} = 1$. For this case the distribution of the r.v. $1 - (1-q_1)\ldots(1-q_{2k-1})$ is approximated by a beta p.d.f. In spite of the attractiveness of this procedure, it should be realized that to date there have been assertions, but there is no published documentation of the robustness of the approximation for the low error rate, highly leptokurtic beta distributions encountered in an auditing model.

The above beta p.d.f. approximation can also be used with parallel processing function (e) of table 5.2.1. For this case when $p_2 = p_4 = 0$ it follows that $P_p = P^2 p_1 p_3$ and hence

$$Q_p = 1 - P_p = 1 - (1-Q)^2(1-q_1)(1-q_3).$$

In general for k parallel processing steps without the possibility of correcting prior errors

$$Q_p = 1 - (1-Q)^k(1-q_1)(1-q_3) \ \ldots \ (1-q_{2k-1})$$

Now when $q_1, q_3, \ldots, q_{2k-1}$ are all beta r.vs., the r.v. $(1-q_1)(1-q_3) \ldots (1-q_{2k-1})$ can be approximated by a beta r.v. Thus rather than evaluating 5.3(8) of table 5.3.1, it follows that the n^{th} moment is given by

$$E(Q_p^n) = E(1-P_p)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i E(1-Q)^{ki} E(q_\beta^i)$$

where q_β represents the approximating beta r.v.

5.5 Consolidated Internal Control System Moments and Their Use

In the previous two sections a set of auditing functions have been defined, and formulas have been derived for calculating summary moments for each function. A method for aggregating the summary moments from a series of auditing functions from an i.c.s. is now considered.

As mentioned in paragraph 4.2.1 an i.c.s. can be viewed as a directed network which passes vectors of probabilities from node to node. In this conceptualization each intermediate link between nodes represents an accounting function that operates on the vectors. It is discussed in this section how this network structure carries over to the probability moments when each link represents an auditing function. This network structure allows the probability moments of a series of auditing functions or a complete i.c.s. to be iteratively calculated using the moment functions of table 5.3.1.

While it would be an unbearably complex task to write out an algebraic expression for the aggregate moments of a series of auditing functions a recursive numerical procedure is quite straightword. Thus, for example equations 5.3(1) and 5.3(2) give the moments of the output error rate r.v. as a function of the input moments and the processing step moments. These output error state moments can now be used as input moments for the moment function of the next auditing function. Starting with the input documents of an i.c.s. a series of moment functions taken from table 5.3.1 can be used to progressively determine the moments of the aggregate error rate r.v. of the i.c.s.

Since these functions can be preprogrammed in subroutine form this recursive procedure is ideally suited for computer evaluation. In application the auditor must describe the document flow network to be analyzed, the auditing functions used for each link, and the moments of the p.d.fs. of the component r.vs. that make up each auditing function. Now through a recursive evaluation the moments associated with the aggregate error rate r.v. can be determined. In the next chapter it is discussed how these moments can be used to develop an orthogonal polynomial approximation to the p.d.f. of the aggregate error rate the moments represent.

The recursive structure of the analysis can be used to reduce the amount of empirical evaluations necessary when conducting a sensitivity analysis. This is useful in studying the implications of variations in the auditor's judgments or the value of potential sample evidence. Thus, if all the intermediate moments are saved during the initial processing, a given sensitivity analysis need not recalculate the intermediate moments prior to the first source of variation.

5.6 Concluding Remarks

The moment functions of table 5.3.1 can be used to completely model an i.c.s. or to analyze areas of weakness within specific i.c.ss. When a complete analysis is desired it is assumed that the auditor has conducted a comprehensive review of the system of internal controls. Thus, it is assumed that the auditor can develop and test for compliance a flow chart of each i.c.s. being modeled. Beside providing a documentary basis for the usual checklists and questionnaires, a flowchart can clearly show the processing steps applied to each document. This is essential if the methods of this chapter are to be used to model an i.c.s.

When the moment procedures are used to analyze areas of weakness within i.c.ss. these same requirements apply to each weakness. Rather than examining all possible error generating situations this application only focuses on those error situations that the auditor considers potentially critical. It is assumed that there is no value in compounding the analysis by considering secondary errors that are assumed to be subsequently identified and corrected. When independent control totals, serial number controls, etc., lead to well documented and verified error control procedures these assumptions seem reasonable. Chapter 7 develops an example which illustrates this approach.

The number of process functions needed to describe a particular area of weakness is very much a function of the type of i.c.s. and the experience of the personnel involved. An experienced payroll clerk may occasionally be able to identify unauthorized overtime amounts not screened out by line supervisors, but a price or quantity error of an invoice may not be subject to the same informal review. Thus, in a payroll application with weak controls over overtime hours it may be useful to include an auditing function for representing the payroll clerk's informal review of time cards.

When numerous processing steps have a bearing on an error rate the auditor may wish to reduce the data specification requirements of the analysis by utilizing point estimates for particular component probabilities of an auditing function. There are no logical constraints on intermixing point and probabilistic estimates for the error rates of the moment functions. The arbitrariness of this approach can be somewhat mitigated by conducting a sensitivity analysis of the implications on final account balances of variation in the parameter used.

CHAPTER 6

THE INTEGRATION OF INTERNAL CONTROL SYSTEM AND ACCOUNT BALANCE EVIDENCE

6.1 Preface

In the previous chapters procedures were developed for representing the auditor's uncertainty about the reliability of each processing step in a repetitive document flow. It also was shown how these component judgments could be consolidated into a summary judgment of the auditor's uncertainty about journal entry error rates. This chapter discusses in general terms the problem of integrating this error rate information with error size information when several i.c.ss. and account balances are involved. Chapter 7 illustrates how the material of this and the previous chapter can be integrated with current auditing practice.

As defined in section 1.3, an i.c.s. is a set of processing steps which lead to or may potentially lead to journal entries. As a further qualification it was assumed that the a priori probability distribution for an error in a randomly selected document of the i.c.s. is constant. For example, if large dollar items are processed with extra care and therefore have different error probabilities, then they should be considered as part of a separate i.c.s. In general, if the auditor believes that documents processed at a particular location or for a particular set of customers or with particular dollar amounts are exposed to different risk levels, then the processing of these documents should define a logically separate i.c.s.

The problem considered in this chapter is how all these individual i.c.s. journal entry error rates can be consolidated with dollar error size information into an aggregate measure of error amount uncertainty for a particular set of accounts. This analysis corresponds to the error amount determination and error amount consolidation modules of figure 1.5.2.

As a consequence of the previous chapter's discussion it is assumed that the lower order moments are available for the p.d.f. of each i.c.s.'s composite dollar error rate. If through an i.c.s. branching operation more than one type of transaction is generated by an i.c.s., it is assumed that these moments are available for each transaction's error rate.

The general form these transactions might take in a merchandising firm is conveniently summarized by Arens and Loebbecke (1976, pp. 232, 407,436 and 409) using T-accounts. The illustrations of these authors summarize the transaction flows of a sales and collection cycle, a payroll and personnel cycle, an acquisition and payment cycle and an inventory and warehousing cycle. Each of these cycles encompass one or more i.c.s. as the term has been defined in this section. While many of the transactions illustrated by Arens and Loebbecke rarely would require a statistical analysis, the elementary T-account diagrams of these authors are a convenient starting point for an analysis.

For the repetitive transactions that are usually subject to statistical analysis it is to be expected that an error in one account is offset by a single error of equal magnitude in one offsetting account.

More complex offsetting structures can be accommodated by using a weighting function to divide the error offset over several accounts. While the subsequent analysis assumes a single offset, the more general case can be easily worked out using parallel logic. It would even be possible to assume that the weighting coefficient is an r.v.

6.2 The Composition of Error Rate and Error Size Uncertainty for a Single Error Type

The first step in bringing together all the error information affecting accounts is to consolidate the rate and size information for each particular type of error. It is assumed that the auditor is concerned with a low error rate environment and has dichotomized his informed judgment and sample evidence into rate and size evidence.

6.2.1 The Beta-Normal Procedure

These assumptions are discussed by Felix and Grimlund (1977). They give a general procedure for consolidating a transaction error rate p.d.f. with error size information that leads to a total error amount probability distribution. As a result of the special form of this distribution (see A4.1(1)) it is called a beta-normal distribution by Felix and Grimlund.

The beta-normal procedure assumes that a standardized beta p.d.f. describes the auditor's uncertainty about possible values of the error rate. It is also assumed that a normal-gamma 2 p.d.f. describes the auditor's uncertainty about the mean and precision of a normal p.d.f. for possible values of the size of identified errors (see also, Raiffa and Schlaifer 1961, pp. 298-303; Winkler 1972, pp. 181-182). The beta-normal procedure combines these sources of evidence into a marginal distribution for the total error in an account due to a particular type of transaction error.

In sampling the transactions of an i.c.s. for error rates, the size of the identified errors can be used as secondary observations of the assumed normal process for error size. After prior to posterior updating of the rate and size samples the two types of information are then combined using the beta-normal procedure. This approach avoids a number of mathematical difficulties that otherwise arise when the error rates are very small. Appendix 4 summarizes these issues and the mathematics of the beta-normal distribution. The appendix also develops several new analytical properties of the distribution not given by Felix and Grimlund.

The analysis of error rates and their implications on account balances need not be restricted to observable ex post errors. In using the beta-normal procedure an "error" can be defined as for example a potential credit default or inventory item write down. Thus, the auditor may find it convenient to analyze the implications of these "errors" using the same error rate and error size decomposition useful for process errors. In these cases, sampling the population leads to estimates of future events rather than observations of past events. These techniques will be illustrated in chapter 7.

The Bayesian emphasis of the beta-normal procedure allows the auditor to capture his informed judgment resulting from his review of internal controls, his cradle to grave examination of transactions, etc. The beta-normal procedure is also analytically compatible with the analysis of i.c.s. error rates discussed in chapters 4 and 5. This analysis showed how the noncentral moments for an unknown summary error rate p.d.f. could be found.

While these moments are incompatible with the beta-normal requirements, they can be used to develop a mixture or weighted average of standardized beta p.d.fs. that is compatible with the beta-normal analysis. This translation is based upon a truncated form of a Jacobi polynomial orthogonal expansion.

Such orthogonal expansions for a p.d.f. based upon the normal distribution are well known. The possibility of using a beta distribution rather than a normal distribution has been recognized in the literature, but has not been explored in any depth. Appendix 3 reviews these issues and develops new procedures for using a Jacobi expansion based upon beta functions.

Paragraph A3.4.4 discusses a truncated form of the Jacobi expansion based upon a mixture of beta p.d.fs. This mixture of standardized beta distributions defined by A3.4(19) with a = 0, b = 1 can be expressed as

$$f(\rho) = \sum_{i=1}^{n} w_i f_{\beta_i}(\rho) \tag{1}$$

Thus $f(\rho)$ is an error rate p.d.f. given by the indicated weightings of beta p.d.fs. It follows from A4.1(1) that the p.d.f. for the total error amount r.v., π_T, can be expressed as

$$\begin{aligned} f(\pi_T) &\cong \int_0^1 f(\rho) f_N(\pi_T | a\rho, 1/b\rho)\, d\rho = \sum_{i=1}^{n} w_i \int_0^1 f_{\beta_i}(\rho) f_N(\pi_T)\, d\rho \\ &= \sum_{i=1}^{n} w_i f_{\beta_i N}(\pi_T) \end{aligned} \tag{2}$$

where f_N is a normal p.d.f. with mean $a\rho$ and precision $1/b\rho$ and $f_{\beta_i N}$ is a beta-normal p.d.f.

Thus, the mixture of beta p.d.fs. is used in place of the usual single beta p.d.f. given by Felix and Grimlund (1977). The resulting total error amount p.d.f. is then the same mixture or weighted average of component beta-normal p.d.fs. Also, from (1) it immediately follows that the noncentral moments of the total error amount p.d.f. are given by

$$\mu_r' = \sum_{i=1}^{n} w_i \mu_r'(\beta_i N) \qquad (3)$$

In summary, these simple steps used in conjunction with the Jacobi expansion procedures of appendix 3 extend the range of the beta-normal procedure to any error rate p.d.f. for which probability moments can be calculated.

6.2.2 Alternatives to the Beta-Normal Procedure

One of the shortcomings with the beta-normal procedure is the assumption that error sizes are normally distributed. Appendix 5 develops an alternative Poisson-gamma model for the composition of error frequency and size information. A natural conjugate joint density for the skewness and scale parameters of the gamma error size process is determined and used to develop several possible forms of the total error amount distribution.

These procedures are found to be analytically less convenient than those of the beta-normal distribution. In particular the integrals for the noncentral moments are not tractable. The natural conjugate distribution also appears to be somewhat difficult to use. However, the distribution appears to have not been previously discussed in the literature, and its properties could be investigated further.

The Poisson error frequency distribution of the Poisson-gamma model is incompatible with the Jacobi expansion technique. However, the gamma distribution procedure for error sizes could be used in conjunction with a beta distribution or Jacobi expansion for error rates. This possibility combines the attractive features of the beta distribution with the skewness potential of a gamma distribution.

Such a "beta-gamma" model can be developed using equation A5.5(5). This p.d.f., $f(\pi_T|r)$, expresses the total error amount uncertainty for the r.v. π_T given that r errors have occurred. Following the Felix and Grimlund discussion the approximation $\tilde{r} = \chi\tilde{\rho}$ is then used (where χ is the number of transactions and ρ is the unknown error rate). It follows that the total error p.d.f. is given by

$$f(\pi_T) \cong \int_0^1 f_\beta(\rho) f(\pi_T|\chi\rho)\,d\rho \tag{4}$$

where $f_\beta(\rho)$ is either an error rate beta p.d.f. or the mixtures of beta p.d.fs. defined by (1). Equation (4) is deceptively simple. From A5.5(5) it can be seen that (4) is an improper double integral with a very cumbersome integrand. These computation difficulties limit the attractiveness of the beta-gamma approach.

6.2.3 The Transition from Transaction to Subsidiary Account Testing

In sampling for the error rates and error sizes of the beta-normal procedure it is possible that the auditor may wish to shift the sample population from transactions to subsidiary accounts. Thus, if the auditor's evaluation of internal controls and tests of transactions indicates significant error rates he may wish to proceed with direct tests of balances. This change of tactics requires that existing error rate and error size information be respecified in terms of the new metric.

In order to accommodate these requirements it is assumed that the subsidiary account error rate in a direct test of balances is given by the r.v.

$$\tilde{\rho}_A = (N_T/N_A)\tilde{\rho}_T \tag{5}$$

where N_T, N_A and $\tilde{\rho}_T$ are respectively the number of transactions, the number of accounts and the transaction error rate r.v. In using (5) care must be exercised in matching up the definition of a subsidiary account error with the number of transactions that might cause this error. Thus, if designated errors in subsidiary balances can result from transactions occurring in earlier years, the auditor must assure himself that the transaction error rate previously developed is representative of this time frame.

If the p.d.f. of the r.v. $\tilde{\rho}_T$ is specified by its noncentral moments as discussed in chapter 4, then the noncentral moments of $\tilde{\rho}_A$ can be immediately determined. If the p.d.f. of $\tilde{\rho}_T$ is or has been approximated by a standardized beta distribution then the p.d.f. of $\tilde{\rho}_A$ will be an extended beta distribution (see section A2.1) defined on $[0, N_T/N_A]$. In both cases the subsequent prior to posterior analysis for Bernoulli sampling of subsidiary accounts cannot be based upon natural conjugate relationships. When the moments of $\tilde{\rho}_A$ can be conveniently calculated, the procedure given in paragraph A2.5.5 can be used to find posterior moments from prior moments and sample information.

Another possibility is to fit a standardized beta distribution using A2.3(12)/(13) to the prior mean and variance and proceed with a natural conjugate analysis.

A routine procedure can be developed for shifting the focus of analysis from transaction error size to subsidiary account error size, when it is assumed that

$$\tilde{\pi}_A = (N_T/N_A)\tilde{\pi}_T \tag{6}$$

where $\tilde{\pi}_A$ and $\tilde{\pi}_T$ are r.vs. expressing the size of identified subsidiary and transaction errors. Defining $k = N_T/N_A$ and assuming that $\tilde{\pi}_T$ has p.d.f. $f_N(\pi_T|\mu,h)$ it is easily shown that $\tilde{\pi}_A$ is also normally distributed with p.d.f.

$$f_N(\pi_A|\bar{\mu},\bar{h}) \tag{7}$$

where $\bar{\mu} = k\mu$ and $h = \bar{h}/k^2$ are the mean and precision of the normal distribution. Further, if (μ,h) has a normal-gamma 2 joint density $f_N(\mu|m,h)f_{\gamma 2}(h|v,\nu)$ then the joint p.d.f. of $(\bar{\mu},\bar{h})$ is also easily shown to be a normal-gamma 2 p.d.f. (Raiffa and Schlaifer 1961, p. 300)

$$f_N(\bar{\mu}|km,\bar{h})f_{\gamma_2}(\bar{h}|vk^2,\nu) \tag{8}$$

The assumptions expressed by (5) and (6) do not follow immediately from the redefinition of the population. It is true that $\rho_A = (N_T\rho_T)/N_A$, however, the equivalent result for the r.vs. is not necessarily true. For both error rates and error sizes it has been assumed that the error uncertainty at the aggregate account level represents a scaling up of the corresponding experience at the transaction level. These are more conservative formulations than assuming that transaction errors may interact and reduce the intensity of the effects on subsidiary accounts. Thus, if $k = N_T/N_A$ is integer valued it could be assumed that $\tilde{\pi}_A = \sum_{i=1}^{k} \tilde{\pi}_{T,i}$. The precision of $\tilde{\pi}_A$ is then h/k or k times as high as the precision given by (7). For this alternative formulation the transformed normal-gamma 2 p.d.f. is no longer of the same normal-gamma 2 form as previously was the case in (8).

As an alternative to the above mathematical formulations for changing to a subsidiary account sample frame, the prior distribution for the size of errors in subsidiary accounts can be subjectively specified. This may be a particularly reasonable approach when the previously identified transaction errors have led to a comprehensive analysis of all the error sizes in each affected subsidiary account.

6.3 Multiple Error Consolidation

When the auditor has identified several error situations each with error rates and error size uncertainty, the methods used to consolidate this uncertainty will vary according to the specific circumstances. Two general approaches are possible. First it may be reasonable to consolidate error rates prior to beta-normal processing. Or second, several beta-normal distributions or mixtures of beta-normal p.d.fs. may be consolidated. These two alternatives correspond to the error rate consolidation and error amount consolidation modules of figure 1.5.1.

6.3.1 Error Rate Consolidation

When several different error rate p.d.fs. affect the same accounts the auditor may wish to consolidate these error rates and only consider one error size p.d.f. This type of consolidation might arise when separate divisions generate different error rates, but there is no reason to expect that the corresponding error size distributions are significantly different. It is assumed that the number of transactions or subsidiary accounts for each division can be used to weight the respective error rate r.vs. Thus

$$\rho_c = \sum_{i=1}^{n} w_i \rho_i \tag{1}$$

where ρ_i is a divisional error rate r.v. and w_i is the divisional weight.

Assuming statistical independence between the component r.vs., the probability moments of ρ_c can be determined from the component moments of each ρ_i. If a particular ρ_i has a beta p.d.f. its moments can be calculated using A2.2(1). Using these component moments, it follows from the multinomial theorem that the r^{th} moment of (1) is

$$\mu_r'(\rho_c) = \sum \frac{r!}{a_1! \ldots a_n!}[w_1^{a_1} E(\rho_1^{a_1}) \ldots w_n^{a_n} E(\rho_n^{a_n})] \tag{2}$$

where a_k is an integer and the summation is over all "a" values such that $\sum_{k=1}^{n} a_k = r$.

Equation (2) can be expressed in a more convenient form for computer evaluation through repeated application of the binomial theorem. Using this approach the r^{th} moment of (1) is found to be

$$\mu_r'(\rho_c) = \sum_{k_1=0}^{r} \sum_{k_2=0}^{r-k_1} \cdots \sum_{k_{n-1}=0}^{r-k_1\cdots-k_{n-2}} \binom{r}{k_1}\binom{r-k_1}{k_2}\cdots\binom{r-k_1\cdots-k_{n-2}}{k_{n-1}}$$

$$\cdot\,[w_1^{k_1}E(\rho_1^{k_1})\cdots w_{n-1}^{k_{n-1}}E(\rho_{n-1}^{k_{n-1}}) \cdot w_n^{r-k_1\cdots-k_{n-1}}E(\rho_n^{r-k_1\cdots-k_{n-1}})] \quad (3)$$

If the r.vs. ρ_i for $i = 1,\ldots,n$ are statistically dependent then

$$w_1^{k_1}\cdots w_{n-1}^{k_{n-1}}\, w_n^{r-k_1\cdots-k_{n-1}}E(\rho_1^{k_1}\cdots\rho_{n-1}^{k_{n-1}}\rho_n^{r-k_1\cdots-k_{n-1}})$$

replaces the [...] term of (3). Procedures for approximating such joint moments are discussed in section A1.4.

Using the moments of (1) determined by (3) the Jacobi expansion discussed in paragraph 6.2.1 can be used to approximate the consolidated error rate p.d.f. as a mixture of beta p.d.f. The subsequent analysis then follows as if consolidation had not taken place.

6.3.2 Circumstances for Beta-Normal Consolidation

Interest in consolidating several beta-normal distributions can arise in two very different ways. In the simplest case a separate beta-normal analysis has been performed for each dollar level stratum of an account or for each division of a firm processing transactions into the account. This process is easily envisioned for a receivables account where different divisions, dollar levels of activity, or customers may lead to different expectations.

When a separate procedure is used to analyze each of several internal control weaknesses there can be an error amount overlapping in particular accounts. This leads to a second need to consolidate several beta-normal distributions. For example, one unknown error amount, e_1, may lead to a debit to accounts payable and a credit to inventory. Another error amount r.v., e_2, may lead to a debit to inventory and a credit to cost of goods sold. These circumstances lead to three error adjustment r.vs.: $-e_1$ to accounts payable balance, $e_2 - e_1$ to inventory balance, and $-e_2$ to the cost of goods sold balance.

The error adjustment r.vs. for summary measures such as total assets, net income, etc., lead to more extensive sums and differences of r.vs. Assume, for example, that there also was an adjustment, e_3, to both accounts receivable and sales. The current assets error adjustment r.v. would then be $e_2 - e_1 + e_3$, the current liabilities error adjustment r.v. would be $-e_1$, and the net income error adjustment r.v. would be $e_3 + e_2$.

These examples illustrate the mathematical problem to be resolved. How can the sum and difference of beta-normal r.vs. (or mixtures of beta-normal r.vs.) be consolidated into one summary p.d.f.? Monte Carlo simulation can, of course, be used. However, for the reason discussed in paragraph 4.2.3 this alternative is not particularly attractive.

In the usual auditing environment the analysis that leads up to the consolidation of beta-normal distributions is based upon a number of informed judgments and modeling assumptions. Consequently, there usually will be some latitude in the required precision of the summary r.vs. The subsequent discussion capitalizes on these circumstances and proposes several alternative mathematical approaches for consolidating a linear function of beta-normal r.vs.

6.3.3 Calculation of Moments for Approximating a Sum or Difference of Beta-Normals

The problem that is now considered is to find the p.d.f. of the r.v.

$$e_T = \sum_{i=1}^{n} w_i e_i \tag{4}$$

where the p.d.f. of e_i is either a beta-normal or a mixture of beta-normal p.d.fs. and $w_i = \pm 1$. Three alternative procedures are discussed in the next paragraph for approximating the p.d.f. of (4). The first two of these, a Jacobi polynomial orthogonal expansion and an extended beta distribution, are based upon probability moments. Consequently, procedures for determining the moments of (4) are first discussed. Then in the next paragraph, the approximation methods are examined.

The noncentral moments of e_T must be computed in order to use the Jacobi expansion approach. This can be accomplished using equation (3) with ρ_c and ρ_i now representing e_T and e_i. As was indicated, statistical dependence can be considered when using (3). The individual noncentral moments of each e_i can be found using A4.2(6). If e_i is a mixture of beta-normal distributions 6.2(3) must be used in conjunction with A4.2(6).

When n, the number of error p.d.fs. to be consolidated, is large it may be more efficient to find higher order moments using the linear operator cumulant property (see for example, Frazer 1958, p. 111, or Johnson and Kotz 1969, pp. 20-21) namely that

$$\kappa(\sum e_i) = \sum \kappa(e_i) \tag{5}$$

In using this procedure the cumulants $\kappa(e_i)$ are calculated from the moments. Then after cumulant summation, either the central or noncentral moments are calculated according to whether an extended beta distribution or a Jacobi expansion approximation is being used. The appropriate formulas for converting between cumulants and moments are given by Kendall and Stuart (1958, pp. 68-71).

In order to use (5) when $w_i = -1$ the preliminary calculation of the beta-normal noncentral moments must be modified slightly. It is easily seen that the effect of a change of variable $\pi_T' = -\pi_T$ to the beta-normal p.d.f. A4.1(1) is to replace the constant $a = \chi m$ by $a = -\chi m$ in A4.1(2). With this slight modification to A4.2(6) for $w_i = -1$, e_T can be expressed as a strictly positive sum of r.vs. with known moments. Equation (5) can then be used.

If n is large and many higher order moments are required it would be convenient to calculate initially the component cumulants $\kappa(e_i)$, rather than just finding the noncentral moments and converting them to cumulants. Unfortunately, the cumulant generating function for the beta-normal distribution is not tractable. It is, however, possible to approximate the beta-normal distribution by an extended beta distribution as discussed in section A4.3 and then find the cumulants of the approximating distribution. Section A2.4 derives a recursive relationship for the cumulants of the extended beta distribution that is very easy to use.

When $w_i = -1$ a change of variable can be applied to the approximating extended beta distribution so that (5) can be used. Paragraph A2.5.6 shows that if the r.v. e_i has an extended beta distribution then the transformed variable $-e_i$ also has an extended beta distribution.

As discussed in paragraph A4.3.2 the extended beta approximation to the beta-normal may be very accurate. Thus, if the cumulants of the extended beta distribution are used in place of beta-normal cumulants in (5), the summary cumulants also should be very accurate. A result due to Hartley (1948), can be used to determine the maximum error caused by this procedure. Let E_i be the cumulative density function error due to the i^{th} beta normal approximation. Letting $\max(e_T)$ be the maximum total error, it follows that

$$\max(e_T) \leq \sum_{i=1}^{n} \max|E_i|$$

If the p.d.f. of e_i is a mixture of beta-normal distributions it is not clear how appropriate a cumulant approximation based upon a single extended beta distribution is. It might be thought that it would be useful to approximate each component of the mixture by an extended beta distribution. However, this approach does not lead to a direct calculation of the cumulants, since in general the cumulants of a mixture of p.d.fs. is not a mixture of the cumulants. For such mixtures of beta-normal p.d.fs. a component by component extended beta approximation can be used in conjunction with a moment recursive calculation. Paragraph A2.4.3 gives the appropriate relationships. Several other recursive relationships that may be useful in this work are also given in the same appendix.

6.3.4 Approximating Procedures for Beta Normal Consolidation

As discussed in paragraph A4.3.1 for the typically large values of χ that are encountered in auditing statistical applications the skewness and kurtosis of the beta-normal p.d.f. corresponds to the corresponding values of the component beta distribution. Consequently, the p.d.f. of a sum of beta-normal r.vs. takes on the general shape of a sum of beta r.vs. Since in the usual auditing environment these distributions are very skewed and leptokurtic, one expects for at least $n \leq 15$ that the summary distribution will also be moderately skewed and leptokurtic.*

In accordance with the discussion of paragraph A3.2.2 these considerations suggest that a Jacobi orthogonal expansion may be an appropriate means of approximating a sum of beta-normal p.d.fs. Further, for any truncated version of the unknown summary p.d.f. defined over $(-\infty,\infty)$, the results discussed in paragraph A3.4.2 imply that the Jacobi orthogonal expansion will be uniformly convergent. This property makes the Jacobi orthogonal expansion a particularly appealing choice in the more general case of both sums and differences of beta-normal p.d.fs. (or mixtures of beta-normals). In this case, the general form of the unknown p.d.f. is not as predictable.

Sections A3.3 through A3.5 discuss the theory and use of the Jacobi orthogonal expansion for approximating a p.d.f. defined over a finite interval. This interval can be determined by truncating the insignificant tails of each beta-normal r.v. of (4) and determining the corresponding interval of probability mass for the total error r.v., e_T. While the required truncation may seem arbitrary, as a practical matter it must be dealt with in all numerical work with improper integrals. The number of probability

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*The convergence to normality of the sample mean (and hence for sums of identical distributed r.vs.) has been extensively investigated using modern simulation technology by Bradley (1973). He concluded that the rapidity of the central limit effect has been greatly exaggerated. For skewed populations he found the central limit effect to be very sluggish, especially over the tail areas.

moments required to fit "adequately" the unknown p.d.f. over the defined interval is difficult to predict. When higher moments are required the previously discussed cumulant and recursive procedures are particularly appropriate.

As an alternative to approximating e_T with a Jacobi expansion, the first four moments can be used to fit an approximating distribution. The well known systems by Johnson or by Pearson might be used (see Johnson and Kotz 1970a, pp. 9-33; Hahn and Shapiro 1967, pp. 198-224; Craig 1936; and Elderton and Johnson 1969). A particularly appealing alternative to these moment approaches or a Jacobi expansion is to fit an extended beta distribution to the unknown p.d.f.

The main advantage of this expediency is the ease with which the approximating distribution can be determined and utilized (see A2.3(4)). However, with a difference of r.vs. there is less reason to expect a robust approximation. It is known (Von Mises 1964, pp. 384-386) that the cumulative density function of the unknown and approximated distributions will have at least as many points of intersection as moments are used in the approximation. Thus, with an extended beta approximation there are at least four points of intersection.

6.4 Concluding Remarks

This chapter has shown how a transition can be formed from internal control error rate evidence to financial statement error adjustments. As discussed in section 1.3, this is as far as the theoretical analysis of this dissertation will proceed. This methodology for developing p.d.fs. for the total error amount in account balances opens up the possibility of a subsequent decision theoretic analysis with sample size and action space considerations. Another interesting question is how a fixed sample budget should be allocated over various compliance and account balance testing procedures. Since the logic developed in this dissertation brings together all these sources of evidence an optimal allocation scheme is in theory obtainable.

CHAPTER 7

IMPLEMENTATION CONSIDERATIONS AND A CASE STUDY

7.1 Preface

While the procedures developed in chapters 4 and 5 could be used by an auditor to comprehensively specify his error rate uncertainty for each processing step of an i.c.s., this is not envisioned as a typical application. Rather, it is envisioned that the auditing functions and integration techniques of chapters 4, 5 and 6 can be combined with existing auditing practices. They then can be used to study isolated weaknesses of i.c.ss. and to appraise their effects on account balances. A case study is developed in this chapter to illustrate this process.

The appropriate conditions for such applications can arise after an auditor's review of internal controls and his initial tests of compliance with the prescribed procedures. Through this review and testing process the auditor begins to form or update his judgment as to where the strengths and weaknesses of the system are. If the control procedures for a material processing step are completely lacking or highly unreliable the auditor may decide to not rely on internal control evidence. Or, with highly reliable controls no further analysis may be necessary. It is the middle ground between no controls and very reliable controls that seems best suited for an evidential integration analysis. These are the conditions under which the auditor may wish to explore in depth the implications of his uncertainty.

This emphasis on examining the implications of uncertainty is largely overlooked in auditing texts. Rather, it is implied that the auditor will either eliminate his adverse uncertainty through additional testing or will not rely upon evidence derived from his study of i.c.ss. There appears to be a need for procedures acceptable to auditors which explore the implications on account balances of this uncertainty. Consequently, the auditor would not be immediately forced to initiate testing procedures to resolve this uncertainty.

The outcome of even exhaustive testing may not lead to a clear choice between high assurance and abandonment of i.c.s. related evidence. For instance, a very tight posterior distribution on a slight to moderate error rate is a source of evidence. However, with the current technology the auditor cannot document the implication pertaining to account balances that he draws from this evidence.

These issues have not been satisfactorily resolved by the decision theoretic emphasis of the theoretical auditing literature. Typically a two state material error model is developed with the corresponding opportunity losses for each i.c.s. state. However, losses arise out of misstated financial statements, not directly out of weak i.c.ss. The theoretical approach assumes that the auditor can intuitively make this transition from actual accounting events to i.c.s. loss states. While the auditor may be able to make this transition, he is currently not able to document the basis of his action.

The implications of a weak internal control are particularly important when it is but one of several isolated areas of weakness. The aggregate effect of all these problem areas on account balances must be examined jointly. Some of the effects may be resolved through a comprehensive account analysis. Where transaction volumes prohibit this census approach, the techniques of chapters 4, 5 and 6 can be used.

In using the auditing functions to develop such an evidential integration model a clear distinction must be made between the management objectives of internal auditors and the financial position objectives of external auditors. For instance, an external auditor may determine that the controls over invoice pricing is at times very weak, but that these errors are nonrecoverable. Consequently the stated sales revenue would not be materially affected by knowledge of the losses. Thus the external auditor's concern hinges on the materiality of other accounting effects of the error.

While the external auditor may not be interested in investigating these losses (see for example, Arens and Loebbecke 1976, pp. 157-160), an internal auditor or management consultant might be able to use the techniques illustrated in this chapter to evaluate some of the potential benefits from implementing costly controls. When an external auditor does investigate these losses, he will not wish to integrate the analysis with other inquiries that may lead to adjustments to financial statements.

The case study presented in this chapter illustrates this distinction between integrative and nonintegrative errors. Six areas of i.c.s. weakness are analyzed using the techniques of chapters 4 and 5. Four of these weaknesses are summarized into an evidential integration model. The remaining two are individually analyzed.

7.2 The Electroplum Case: The Problem

In the remainder of this chapter some of the techniques developed in chapters 4, 5 and 6 are illustrated in a fictitious case study. In a specific auditing engagement it is to be expected that only a small subset of the methods developed in these chapters will be needed. The full set of techniques are, of course, necessary in order to accommodate a variety of circumstances. The following case study conforms to this general pattern.

7.2.1 The Electroplum Scenario

Electroplum, an electrical and plumbing building materials supplier in a large metropolitan area, had always been a very routine audit assignment. However, about three years ago, with the passing away of the founder of the firm, a major block of stock changed hands. This led to new leadership and a broadening of both the debt and equity bases of the firm to support expansion. A line of heating supplies had been added, and then one-and-one-half years ago Electroplum opened a large wholesale outlet in a distant location experiencing rapid economic growth. With an unanticipated escalation in the interest rate the building boom had collapsed early in the fiscal year, leaving Electroplum seriously over-extended and not financially strong.

Management had reacted by cutting way back on personnel and relaxing credit policies to promote sales. As a result of the general confusion in adapting to the new mode of operation and the shortage of staff personnel, a number of internal control problems had developed over the last year. At the auditor's year-end review, six weeks after closing, the following six unresolved areas of weakness had been observed. These problems had been mainly identified during the interim review of internal controls and tests of these controls for compliance with the prescribed procedures.

7.2.2 The Sales and Collection Internal Control System

7.2.2.1 Credit Policy

Early in the fiscal year the credit policy had been changed to allow a $2,500 initial credit limit for each new customer. This credit restriction was then removed or raised when a satisfactory credit check was received. During the interim review, two months before closing, a sample of 50 credit applications for the metropolitan area had been compared with the accounts receivable records for these new customers. Only three compliance errors had been found where the credit policy had been ignored.

At the branch a similar sample had found 28 cases where unlimited credit was being extended without proper justification. From the audit reports on file the auditor had estimated that about 40% of these compliance errors could lead eventually to credit losses. At the time of the interim review these results were reported to management. Currently the percentage of unpaid balances 60 days or over for the metropolitan area was up by 20%. At the branch the 60 day and over percentage was three times as high as in the metropolitan area.

Management was aware of the deterioration in the accounts receivable position, but considered it a temporary result of the economic downturn that would clear up without serious losses. This attitude, plus management's extreme sensitivity to profit reducing adjustments placed the auditor in a very difficult situation. Rather than risk losing or antagonizing the client with arbitrary proposals, the auditor felt it would be best to investigate further before taking any action.

It was relatively easy to isolate high risk accounts, but estimating the size of the potential losses was much more difficult. The auditor was considering a further sample from all new accounts at the metropolitan areas, and all accounts at the branch. This could be used to construct an estimated lack of compliance and default rates for each of the two populations. A subsample of these high risk accounts could then be examined in detail to estimate the size of the anticipated losses in each population. The auditor felt that the current allowance for uncollectibles was adequate to cover the lower risk accounts with satisfactory credit reports. Consequently, the statistical analysis would only focus on the lack of compliance accounts.

7.2.2.2 Unrecorded Sales

During the interim review the auditor had uncovered a difficulty in the internal controls for sales and collection at the branch. This had developed out of a new marketing strategy introduced with the opening of the branch.

In order to develop this new marketing area the firm had hired four field salesmen known to many local contractors. A standard sales lead was to call up contractors after work and ask them if there was anything they desperately needed that the salesman could pick up at the branch and bring out to the job site the next day. Consequently, the field salesmen were often in the warehouse filling small orders to take with them.

Rather than taking the time to fill out a shipping order, the salesmen were allowed to drop their order forms in the in-basket used for mailed and phoned in customer orders. When the order clerks came across these orders they were to fill out a shipping document, enter the shipping document number on the salesman's order form and pass it on to another basket for eventual filing in the customer file. An identical procedure was used for regular customer orders.

In reviewing the customer order file, the auditor had found that only about 80% of the orders had a recorded shipping document number. In a sample of 30 customer orders without shipping numbers it was possible in each case to identify a valid shipping document. For an equivalent sample of salesmen orders, no record of a shipping document was found in 18 cases. Later, after management had investigated the matter, the auditor was quietly told that management had concluded that an informal system of "free gifts" had developed to encourage sales. The practice had been quietly stopped at year end with tighter controls, but it had been decided not to conduct a detailed appraisal of these unrecoverable losses.

The auditor's review of the new internal controls, six weeks after year end, indeed verified management's assertion that the practice had been stopped. However, the auditor was still concerned about the unknown scope of the practice. It might be possible that the branch sales were materially inflated by attracting customers through an unreported give-away policy.

7.2.3 The Inventory and Warehousing Internal Control System

7.2.3.1 Obsolete Inventory

During the last audit a number of obsolete electrical items no longer meeting the local building code in the metropolitan area had been observed in the main warehouse inventory. Management had stated at that time that these items were still up to code in a number of locations being considered for a second branch. While writedowns for a metropolitan area firm might be in order, this was not representative of the firm's future business area to which the items would be transferred.

With the deterioration of the firm's position such an expansion step seemed rather remote. Not being familiar with the details of the product line, during the interim review the auditor had taken a sample of 100 electrical stock numbers with cost values between \$10 and \$25 per standard manufacturing purchase unit. The calculated sales for these items indicated that 18% of them were stale items. The average recorded dollar value for each stock number was \$428.

Assuming these figures were representative of the total electrical inventory, the auditor calculated that 16% of the electrical inventory's dollar value was represented by stale items. This amounted to 9% of the total inventory value, from which it was projected that a 5% to 6% writedown of the total inventory was in order.

The interim letter to management had suggested that writedowns seemed to be called for. However, management had reiterated its original position. While a 5% to 6% writedown was not material enough to make a strong issue of this point, the auditor was not very confident in his very tentative calculations.

7.2.3.2 Fabrication of the Inventory Count

According to the branch warehouse manager, the metropolitan warehouse's relaxed policy toward employee pilferage had proved to be unrealistic at the new branch. While no reliable evidence of pilferage had been established, the branch manager had recently implemented strict controls. This had led to some employee friction and suggested to the auditor that there could be some fabrication of the end-of-period inventory to cover up for shrinkages.

A new inventory would be very difficult to take at this late date, but the auditor could use an EDP audit program in conjunction with the firm's automated inventory records to generate a dollar unit sample of the branch's end-of-period inventory. Management had reluctantly agreed to assign an experienced employee to investigate and document for the auditor's review a sample of 100 stock numbers. From this information the auditor could determine the error amount per stock number dollar for each of the stock numbers in the sample. He could then construct sample error rates and error sizes for each of the sampled dollars.

7.2.4 The Acquisition and Payment Internal Control System Shipping Charges

Verifying the freight charges on shipments from manufacturers had always been a troublesome weakness in the payment i.c.s. A management study three years previous had concluded that about 16% of the rail and trucking bills and about 6% of the remaining bills contained overcharges. Either the quoted shipping weights were in error or the appropriate tariffs were not applied. The accounting department only reviewed the rail and trucking bills, since the dollar error amounts of the remaining bills were usually minor, and often not recoverable.

The Interstate Commerce Commission regulations pertaining to the rail and trucking rates were difficult to follow, and it was not always possible to verify the quoted shipping weights. Recognizing that exhaustive error checking was not feasible the following review procedure had been devised. The rail and trucking bills first were collected together. Then all the rail and trucking bills over a specific dollar amount were culled out for review. This led to a "census" stratum. An experienced accounts payable clerk then picked out of the remaining rail and trucking bills those that in his judgment should be investigated.

At the time of the management study three years previous, it had been estimated that about 3% of the rail and trucking bills were overlooked and not included in the error review procedure. Of the rail and trucking bills with overcharges, it was estimated that 25% would not be examined, 55% would be picked out by the judgmental process and 20% would have large dollar amounts and be automatically reviewed. It was also concluded that the error analysis of the 55% group would only pick up about 90% of the errors in this group. The large dollar amount census stratum was carefully examined and rarely, if ever, led to an oversight. Subsequent experience with the system had shown that 10% of all the bills fell into the census stratum, 20% were optionally examined and the remaining 70% were not looked at.

Up until the opening of the branch the system had apparently worked satisfactorily. Since then, the stocking of the new branch had led to a lot of unfamiliar types of shipments. Further, the experienced payables clerk who did the sorting had been promoted to another position at the branch. In the auditor's judgment, there was reason to believe that the residual rail and truck overcharge rate was significantly higher than 5.21% [.03(16%) + .97(16%)(.25 + .55 x .10)].

7.2.5 The Payroll Internal Control System Overtime Payments

In order to promote sales at the new branch Electroplum extended the normal wholesale business hours to 7 p.m. Staggered shifts and a 10 hour, 4 day workweek were selectively introduced on an experimental basis. According to the agreement worked out with the union, no employee could be forced to accept a 4 day workweek and Electroplum reserved the right to switch 4 day employees back to a 5 day schedule. There had been as a result of these provisions considerable switching back and forth.

These complications, plus the extensive reduction in employment, had led to considerable overtime for the remaining warehouse personnel. Five day a week employees sometimes worked 10 hours. Four day a week employees sometimes worked part of an extra day of overtime or on regular time to make up for leaving early (another union provision). Because of the high fringe benefit costs for each additional employee and the general economic uncertainty, branch managers judged that the current system with overtime and flexibility was more advisable than increasing the staff.

There was one administrative problem which the auditor had detected during the interim review. As required by the union contract, when overtime was utilized, the employee had to be paid for at least 2 hours of overtime. Thus, many 10 hour days were clocked in and it was not clear from the time card hours themselves how much, if any, overtime hours were represented by a given Thursday to Wednesday pattern of hours.

Each supervisor was supposed to review all the time cards for his employees and indicate on the card the regular and overtime hour breakdown. The overtime hours were then approved by the foreman and sent to the payroll clerk. While strong controls were not built into the system, the auditor had concluded at the interim review that the system was working fairly well in the electrical department.

In the plumbing and heating department, the auditor was uncertain as to the extent of the controls. Rather than verify the overtime hours using employee records, the manager of this department had just glanced over the cards. Apparently very few errors had been detected and without this feedback the supervisors had grown accustomed to relying on the overtime hour breakdown that the employees often entered on the card.

The auditor had pointed out this weakness to central management at the time of the interim review. Statistical testing had not been proposed since the possible overpayments were not recoverable, and this was more a matter of administrative control than financial statement integrity. Corporate management had stated at the time that they would handle the matter internally.

In later discussions the branch manager had convinced corporate management that the personnel situation with the plumbing and heating foreman was very delicate. Management had decided to hold off until after the autumn heating business, and then let the auditor look into the matter. Rather than just looking at the plumbing and heating department it was suggested that all overtime processing procedures at the branch be examined.

7.2.6 The Auditor's Problem

It should be emphasized that the six internal control weaknesses postulated for Electroplum represent only those areas for which it is not feasible to conduct an exhaustive census. From the statistics available to the auditor, it appears that none of the problems are extremely critical. However, there are two further issues to be considered. What is the joint effect, and how is this effect influenced by the uncertainty of the auditor toward the stated statistics?

The auditor is, of course, subject to both economic and time pressures to complete the audit. In order to justify more complete testing it may be useful to explore first with analytical techniques the implications of his prior judgment. This can be accomplished by using the auditing functions and related techniques to develop an auditing model based upon prior p.d.fs. If after this analysis it is decided to conduct additional tests it is imperative that the significance of the results be fully investigated. After prior to posterior analyses the auditing model again can be used to explore the implications of this new set of data.

As will be seen, only a few of the techniques developed in chapters 4, 5 and 6 are used in this example. However, as previously stated, since the requirements of each audit are usually different, the flexibility of a full set of techniques is necessary.

7.3 The Electroplum Case: The Analysis

In this section the Electroplum scenario is interpreted in terms of the statistical techniques developed in the previous chapters. The implications of the six unresolved problems are summarized, and then it is explained how these techniques can be used to develop a model of the auditor's total error uncertainty. In order to facilitate easy reference to the complementary material of section 7.2, the paragraph numbers of this section have been aligned with those of section 7.2.

7.3.1 Error Implications

The first step in the development of the Electroplum model is to summarize the types of accounting adjustments that might arise from the weaknesses. Table 7.3.1 presents these results. As discussed in paragraph 7.2.2.2, it has been assumed that the unrecorded sales are uncollectible, but are still of interest to the auditor. At management's request the excessive overtime payments will also be investigated. Each of the error amounts (designated by e_{xy}) is an uncertain amount or r.v. The resulting adjustments to the income statement and balance sheet accounts are presented in table 7.3.2. This analysis indicates the two analytical tasks yet to be performed. Each r.v., e_{xy}, must be specified and the p.d.f. of functions of these r.vs. must be determined.

The following analysis for each i.c.s. weakness can be based upon either a noninformative prior judgment or a more specific informed judgment arising in part from the statistics reported in section 7.2. In the latter case, the techniques to be discussed can be used prior to additional sampling to explore the implications of the prior judgment and potential sample evidence.

Table 7.3.1

Adjustment Transactions for the Error Analysis

I.	The Sales and Collections ICS			
	A. Credit Policy	DR Bad Debt Expense	e_{cp}	
		CR Allowances for Uncollectibles		e_{cp}
	B. Unrecorded Sales	e_{us} (No Recoverable Adjustment)		
II.	The Inventory and Warehousing ICS			
	A. Obsolete Inventory	DR Inventory Writedown Expense	e_{oi}	
		CR Inventory		e_{oi}
	B. Fabrication of Count	DR Inventory Loss	e_{fc}	
		CR Inventory		e_{fc}
III.	The Acquisition and Payment ICS			
	A. Shipping Charges	DR Accounts Payable	e_{sc}	
		CR Inventory		e_{sc}
IV.	The Payroll ICS			
	A. Overtime Payments	e_{op} (No Recoverable Adjustment)		

Table 7.3.2

Financial Statement Error Adjustments

I. Income Statement	
Sales	....
Less Cost of Goods Sold	....
Inventory Loss	e_{fc}
Inventory Writedown	e_{oi}
Bad Debt Expense	e_{cp}
Administrative Expense	....
Net Income	$- e_{fc} - e_{oi} - e_{cp}$
II. Balance Sheet	
Assets	
Cash	....
Accounts Receivable (Net of Allowance)	$- e_{cp}$
Inventory	$- e_{sc} - e_{fc} - e_{oi}$
Total Current Assets	$- e_{sc} - e_{fc} - e_{oi} - e_{cp}$
Fixed Assets	....
Total Assets	$- e_{sc} - e_{fc} - e_{oi} - e_{cp}$
Liabilities and Owners Equity	
Accounts Payable	$- e_{sc}$
Long-Term Debt	....
Equity	....
Beginning Retained Earnings	....
Net Income	$- e_{fc} - e_{oi} - e_{cp}$
Total Liabilities and Owners Equity	$- e_{sc} - e_{fc} - e_{oi} - e_{cp}$

7.3.2 Sales and Collection Error Analysis

7.3.2.1 Credit Policy

The credit policy error rate analysis is an application of the compliance error discussion of section 4.3 and in particular 4.3(1). However, it is not necessary in this analysis to make a distinction between the in error (D_e) and not in error (D_{ne}) initial error states as discussed in section 4.3. Consequently, the corresponding conditional probability notation of that section is not used. Further, since the presence of a favorable credit report was noted to provide positive evidence of compliance to policy no distinction must be made between assertive compliance and actual compliance as represented by equation 4.3(2). Also, since the auditor is confident that the current allowance for bad debts is adequate to handle defaults resulting from customers with acceptable credit reports, it is assumed that $P(E|C) = 0$ in 4.3(1). Consequently for each location

$$P(E_{cp}) = P(E|\sim C)P(\sim C) \tag{1}$$

Thus, the analysis of the credit policy only considers lack of compliance "errors" (or defaults). Since there is no reason to expect that the dollar sizes of errors will differ between the two locations, error rate consolidation is used as discussed in paragraph 6.3.1. The firmwide error rate for credit policy default errors is

$$P(E_{cp}) = w_M P_M(E_{cp}) + w_B P_B(E_{cp}) \tag{2}$$

where the component probabilities for the metropolitan and branch locations are given by (1) and $w_M + w_B = 1$ are population percentages easily developed from the sample populations defined in paragraph 7.2.2.1.

For each location a prior to posterior beta p.d.f. analysis based upon the current sample data can be used to develop a p.d.f. for the lack of compliance. The probability of an error given a lack of compliance, $P(E|{\sim}C)$, is not easily specified. From a sample of these compliance errors, the auditor must exercise his judgment and predict for each sample if less than full payment is to be expected. This ex ante predictive analysis is briefly discussed in paragraph 6.2.1. These predictive observations can then be used in a prior to posterior beta p.d.f. analysis for $P(E|{\sim}C)$.

Assuming that a common p.d.f. for $P(E|{\sim}C)$ is used for both locations, it follows from (1) and (2) that

$$P(E_{cp}) = P(E_{cp}|{\sim}C)[w_M P_M({\sim}C) + w_B P_B({\sim}C)] \tag{3}$$

The procedure discussed in paragraph 6.3.1 can now be used to find the noncentral moments of $P(E_{cp})$. The moments for the component beta p.d.fs. are given by A2.2(1). The moments of $P(E_{cp})$ are then used to develop a mixture of beta p.d.fs. as discussed in paragraph 6.2.1.

For those compliance errors, which are predicted to lead to a default, a further prediction of the size of the default can be used in a normal-gamma 2 prior-to-posterior analysis. The beta-normal analysis of paragraph 6.2.1 and in particular equation 6.2(2) is then used to determine the p.d.f. of e_{cp}, the predicted total error (default) amount.

7.3.2.2 Unrecorded Sales

A p.d.f. for the total amount of unrecorded sales can be developed using a beta-normal analysis, without any of the complications that arose in analyzing the credit policy. A sample population of salesmen orders within the customer order file can be defined and sampled. An attempt can then be made to trace each sampled item to a valid shipping order. The amount of the "giveaway" in the sampled error documents is then determined. After prior to posterior updating of the beta and normal-gamma 2 p.d.fs., a beta-normal p.d.f. for the total error amount, e_{us}, can be determined. The currently available statistics discussed in paragraph 7.2.2.2 can, of course, be used in this analysis with or without subsequent samples.

7.3.3 Inventory and Warehousing Analysis

7.3.3.1 Obsolete Inventory

A stratified beta-normal analysis can be developed using the original unit cost of the inventory items. For each stratum a beta-normal p.d.f. can be determined using procedures similar to those used for the unrecorded sales, as discussed in paragraph 7.3.2.2. The p.d.f. of the obsolete inventory adjustment error, e_{oi}, is then a weighted sum of the beta-normal p.d.fs. for each stratum. One of the beta-normal consolidation procedures discussed in section 7.3 can be used to approximate the p.d.f. of e_{oi}.

7.3.3.2 Fabrication of the Inventory Count

The details of the proposed dollar unit sample are discussed in paragraph 7.2.3.2. The indicated procedures lead to an error rate of sampled dollars and a prorated error size for the in error dollars. After prior to posterior updating a beta-normal distribution on the total number of fabricated inventory dollars, e_{fc}, can be determined.

7.3.4 Acquisition and Payment Analysis Shipping Charges

Error control processing function (g) of table 5.2.1 can be used to study the freight overcharges. Figure 7.3.1 gives a numerical version of (g) constructed from the statistics given in paragraph 7.2.4.* It is assumed initially that all error probabilities are known quantities rather than r.vs.

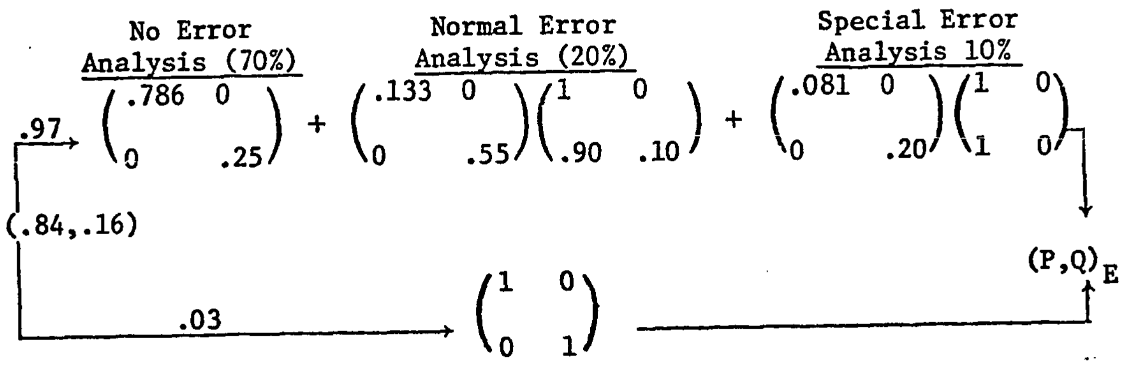

Figure 7.3.1. Accounting Function (g) for Shipping Charges.

The effect of a deterioration in the performance of the error analysis sorting and processing can be studied by replacing .03, .10, .25, .55 and .20 by the r.vs. q_7, q_4, q_2, p_2 and $\bar{p}_2 = 1 - q_2 - p_2$. This is the notation used for function (g) in table 5.2.1. The remaining values are either not relevant or not expected to have been affected by the new types of shipping. They are assumed to be known fixed values.

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*The probability value .786 = [.70-.16(.25)]/.84. Similar logic is used to calculate .133 and .081.

Given a prior beta p.d.f. for each of the r.vs., formula (g) of table 5.3.1 can be used to determine the moments of the output error rate r.v. Q_E. The p.d.f. of q_7 can be easily specified through sampling, however, the p.d.f. of the remaining r.vs. are difficult to determine. Consequently, an ex ante to sampling analysis can be very useful in determining the implications of the auditor's professional judgment about these error rates.

Formula (g) of table 5.3.1 can be simplified using table 5.3.2. From (g) of table 5.2.1 it is easily seen that $q_3 = q_5 = 0$. According to table 5.3.2(g), this implies that the indices j, k and i are fixed at $j = k = 0$ and $i = n$. Applying these simplifications to equation (g) of table 5.3.1 with $Q = .16$ and $q_6 = 1$ yields

$$E(Q_E^n) = (.16)^n \sum_{\ell=0}^{n} \sum_{m=0}^{n-\ell} \sum_{r=0}^{n-\ell} (-1)^m \binom{n}{\ell} \binom{n-\ell}{m} \binom{n-\ell}{r} E(q_7^{\ell+m} q_2^{n-\ell-r} q_4^r q_2^r) \qquad (4)$$

Using table 5.3.2 and the related discussion of paragraph 5.3 the moments of (4) are easily determined when the r.vs. are statistically independent. The moments of the component beta p.d.fs. are given by A2.2(1). Considerations of correlation between these component distributions can be accommodated using the procedures of section A1.4.

Given the moments of (4) the unknown p.d.f. can be approximated as a mixture of beta p.d.fs. as discussed in paragraph 6.2.1. Thus, in order to determine a beta-normal p.d.f. for the shipping charge r.v. it is only necessary to specify a normal-gamma 2 distributions for the recoverable freight adjustments. Assuming that the past adjustments are representative of the possible adjustments, this data can be used to develop the required distribution. However, care must be taken to exclude the adjustments of the large dollar amount "census" stratum in this determination. Finally, since some of the adjustments have already been achieved, the financial statement error adjustment r.v., e_{sc}, is given by a mixture of beta-normal r.v. minus the currently collected amount.

7.3.5 Payroll Analysis Overtime Payments

Using accounting functions (a) and (c) of table 5.2.1 figure 7.3.2 can be constructed for analyzing overtime payment error rates. The subscripts e, h and p are used to indicate the electrical, plumbing and the heating and payroll clerk functions. The additional letters s and f abbreviate the modifiers "supervisors" and "foremen."

Electrical Foreman
(function a)

$$(P,Q)_{es} \longrightarrow \begin{pmatrix} 1 & 0 \\ p_e & q_e \end{pmatrix} \longrightarrow (P,Q)_{ef}$$

Plumbing and
Heating Foreman
(function a)

$$(P,Q)_{hs} \longrightarrow \begin{pmatrix} p_{h1} & q_{h1} \\ p_{h2} & q_{h2} \end{pmatrix} \longrightarrow (P,Q)_{hf}$$

Weighted Average Merging
(function c)

$$(P,Q)_{ef} \xrightarrow{w_e} (P,Q)_f \xleftarrow{w_h} (P,Q)_{hf}$$

Payroll Clerk
(function a)

$$(P,Q)_f \longrightarrow \begin{pmatrix} 1 & 0 \\ p_p & q_p \end{pmatrix} \longrightarrow (P,Q)_p$$

Figure 7.3.2. Functions (a) and (c) for Overtime Payments.

Figure 7.3.2 indicates the flow of time cards from the departmental supervisors to the foremen and then to the payroll clerk. The r.vs. q_e, q_{h_1}, q_{h_2} and q_p represent the probabilities that a foreman or payroll clerk will fail to correct a time card with an invalid distribution between regular and overtime hours. As the matrices indicate it is assumed that no new errors are introduced by the electrical foreman or the payroll clerk since in both of those areas personnel records are consulted before making any modifications to a time card. Even though the payroll clerk has no specific responsibility for verifying all the overtime hours, his familiarity with the process adds another control to the system. Thus q_p is not assumed to be 1.

After a preliminary review of the documented flow the auditor can specify p.d.fs. for each r.v. and conduct a preliminary analysis. If the uncertainty in any Q of figure 7.3.2, or in a beta-normal r.v. incorporating an error rate Q, is unacceptable the auditor may wish to develop further sample evidence. A sample of time cards from each department can be used in a prior to posterior analysis for the r.vs. Q_{es} and Q_{hs}. The identified errors then can be used as sample observations for prior to posterior updating of q_{h_1}, q_{h_2} and q_e. Those errors that are not corrected by the foreman can be used as observations for prior to posterior updating of q_p.

Moment functions (a) and (c) of table 5.3.1 can be used to calculate recursively the moments of the final error rate Q_P. This iterative procedure is discussed in section 5.5. The remaining steps in the analysis do not introduce any new considerations. They are used to determine a p.d.f. for e_{ot}, the total overtime error amount.

7.3.6 Aggregate Error Amounts

The Electroplum analysis of section 7.3 has discussed how a total error amount p.d.f. can be determined for each of the six areas of weakness identified by the auditor. The aggregate uncertainty or cumulative effect of these errors is also of concern to the auditor. As illustrated in table 7.3.2 there are aggregate effects on net income, inventory and total current assets. The total asset effect and the total liabilities and owners equity effect is in this case the same as the total current asset effect.

In analytical terms, aggregate p.d.fs. may be desired for $e_{fc} + e_{oi} + e_{cp}$, $e_{sc} + e_{fc} + e_{oi}$ and $e_{sc} + e_{fc} + e_{oi} + e_{cp}$ where each e_{xx} is a r.v. with either a beta-normal p.d.f. or a weighted sum of beta-normal p.d.fs. This problem has been discussed in paragraphs 6.3.2 through 6.3.4. Several approximations to such aggregate p.d.fs. are suggested in paragraph 6.3.4.

The particular approximation an auditor may wish to use is very much a function of the computing capabilities he may have available. Once the appropriate subroutines have been developed, the auditor may wish to conduct initially one very accurate analysis using the Jacobi polynomial expansion approach. Then if an extensive sensitivity analysis is desired the computationally simple extended beta approximation might be considered.

These comments are, of course, strictly conjectures based upon a prior judgment of the empirical implications of the theory developed in this dissertation. While such empirical issues transcend the boundaries of the dissertation, they do emphasize the need for subsequent empirically oriented research.

7.4 Concluding Remarks

The Electroplum scenario and this chapter's analysis has been used to demonstrate how the methodology of this dissertation might eventually be applied to routine types of auditing problems. Besides offering a pedagogical exercise for better comprehending the theory of the previous three chapters, the analysis also serves to highlight some of the possible areas of future research.

The need for computer oriented empirical research was emphasized in the concluding paragraph of the last section. There are additional considerations of the robustness of the various probability models used in the analysis for capturing the auditor's judgment and representing the types of dollar errors that usually occur. While some prior research has been done, there are numerous methodological questions associated with the specification of the auditor's judgment. Existing and possibly new methodology suggestions need to be evaluated in an auditing setting. The impact of education, assessment training and personality characteristics on the performance of individuals and groups of assessors could be researched in greater depth.

CHAPTER 8

CONCLUSION

As stated in chapter 1, the specific research question examined by this dissertation is "How can a quantified form of the auditor's judgment be analytically combined with additional sources of evidential information?" The analysis of chapters 4, 5, and 6 has shown how this can be accomplished. Throughout this analysis p.d.fs. for the auditor's uncertainty about error rates and error amounts are combined with sample evidence and auxiliary a priori information.

In this analysis a method has been presented for recognizing both the intricate control structure of an i.c.s. and the auditor's uncertainty about the reliability of its operation. It has then been shown how the intertwining effects of the reliability of several i.c.ss. can be combined with a direct substantive analysis of the reliability of an account balance. Finally several alternative methodologies have been proposed for aggregating the error uncertainty in several accounts, each of which is subject to the above interplay between i.c.ss. and an account.

The special requirements of this analysis have led to several mathematical statistics studies presented in the appendices. The predominate role played by the beta p.d.f. has led to a survey and analysis of many properties of the extended beta p.d.f. A treatise on the use of Jacobi orthogonal polynomials in approximating a p.d.f. has provided an analytical basis for combining i.c.s. and account balance forms of uncertainty. Finally a Bayesian natural conjugate analysis for a gamma p.d.f. process has been developed. The significance of these mathematical aspects of the dissertation is not necessarily confined to the specific auditing application.

While the scope of this dissertation is fairly broad it should be emphasized that the analysis is subject to the usual limitations of models. Thus it is not clear how well the assumed p.d.f. structure can emulate the actual probabilistic processes and judgmental forms of uncertainty of concern to the auditor.

If the model is considered as a potential descriptive theory of an auditor's implicit evidential integration process, the predictive significance of the model might be used to evaluate the robustness of the analysis. Thus one criterion for evaluating the evidential integration model is how well it can emulate the types of conclusions implicitly reached by auditors. That is, are the summary judgments of the auditor compatible with the analytical judgments that arise from the model. These latter judgments are of course an integration of both objective evidence and auditor's more detailed component judgments.

If the model can be shown to have predictive significance, it might be of interest to focus on costs and benefits and explore its actual utilization. In section 1.2 it has been suggested that, through an accompanying sensitivity analysis, the model might generate new types of benefits and reduce the costs of sampling. As a possible additional benefit, the model allows the auditor to document the process of analysis and narrow the scope of the subjective judgments that others observe in the auditor's methodology.

It would be very difficult to divorce the evaluation of the model from actual auditor performance and deal with the model as a normative methodological theory. Thus, rather than assuming a descriptive objective of emulating actual auditing conclusions, an independent normative evaluating criteria could be constructed. This leads to many difficult questions of error amount materiality.

From such a normative perspective, the analytical conclusions of the model might be compared repeatedly with the actual total error amounts found in a selection of accounts. This is not only very difficult to do, but also very difficult to evaluate. Besides the obvious problems of research design, there are questions as to what type of scoring rule should be used for each comparison (see, for example, Winkler 1967). Thus the p.d.f. for the total error developed by the model must be scored using a single number, the actual total error amount developed through some independent study of each account.

Such normative and descriptive evaluations of the model are intricately connected with the methodology used by the auditor to quantify his judgmental uncertainty. Until such time as this judgmental methodology is independently evaluated in an auditing setting it will be very difficult to separate out the potential inadequacies in the model from those arising from the judgmental specification process. The development of such an auditing methodology for specifying error rate and error amount uncertainty is thus an important direction for future research.

Another possible avenue for future research is to investigate the error size normality assumption of the beta-normal analysis. However, given the infrequency of such errors it is very difficult to sample actual accounting populations for these errors. Thus it is difficult to form judgments about the adequacy of the normality assumption. A more straightforward approach of some value would be to make comparison between the beta-normal analysis and the beta-gamma model considered in paragraph 6.6.2. Such a simulation study could provide some insights as to the implications of possible deviation from normality on the predicted total error uncertainty for an account balance.

In a problem with as many facets as those that arise in the integration of auditing evidence, alternative analytical models are no doubt possible. Others with different dispositions and technical expertise could undoubtedly develop a completely different type of theory for the integration of auditing evidence. For example, the firm's business setting, current economic constraints and risk levels are not explicitly utilized in the model. Rather it is assumed that these factors enter into the auditor's evaluation of his uncertainty about specific error rates and amounts.

Similarly there are no provisions in the model for explicitly utilizing the ARIMA and regression predictions that arise out of an analytical review of account balances (see Deakin and Granof 1974; Kinney and Bailey 1976; and Kinney 1977). Rather it is assumed that these procedures are used in part by the auditor to narrow the focus of his concern, and hence isolate the accounts that will be subject to an integration of evidence analysis.

The analysis developed in this dissertation concludes a theoretical phase in the author's study of the integration of auditing evidence. This is not to suggest that additional theoretical work may not be forthcoming, but rather to emphasize that the next step in this stream of research appears to be empirical.

A major aspect of the dissertation's analysis has been based upon the use of probability moments in a Jacobi orthogonal polynomial series expansion for the form of unknown p.d.fs. As discussed in appendix 3 this type of procedure was originally suggested by Pinney (1947) for a curve fitting application, and has been subsequently largely overlooked and never adequately explored. While the mathematics of the procedure have been developed in more depth in this dissertation, an empirical evaluation of the use of the procedure in an auditing setting remains to be carried out. There are unanswered questions as to the rate of convergence of the expansion which will affect the cost effectiveness of the model when used in an extensive sensitivity analysis.

Such an evaluation could build upon the case study developed in chapter 7. This approach would add a focus and structure to such empirical work. It would also serve a second objective of developing an empirical demonstration of the use of the evidential integration model. At this stage in the development of the model, the author's thinking might profit from the broader exposure possible with an empirical example that practitioners could follow.

Thus it is seen from the discussion of this concluding chapter that the next step in the development of the evidential integration model is twofold. There is a need for both empirical simulation research and research directed at perfecting the auditor's judgmental specification process. It is the author's intention to proceed immediately with the first research objective and work with others with a behavioral research background on this latter objective.

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APPENDIX 1

NOTATION AND MISCELLANEOUS RESULTS

A1.1 Preface

Section A1.2 of this appendix defines the notational conventions used in this dissertation. The subsequent two sections discuss some miscellaneous mathematical results. In section A1.3, it is shown how the reliability analysis of Cushing (1974) can be restated in the Yu and Neter (1973) format. Section A1.4 discusses some multivariate probability models that may be useful in calculating the joint moments of statistical dependent random variables.

A1.2 Notational Conventions

In paragraph A1.2.1 the technical abbreviations and numbering system used in the dissertation are defined. This is followed by a discussion in paragraph A1.2.1 of the mathematical notation used to specify probability density functions and probability moments.

A1.2.1 Technical Abbreviations and Numbering System

As is currently the custom in technical writing this dissertation avoids using a variety of abbreviations. However, several technical forms, that are used repeatedly, are abbreviated in order to improve the exposition. These are:

r.v. = random variable
l.h.s. = left-hand side
i.c.s. = internal control system
p.d.f. = probability density function
r.h.s. = right-hand side

Plural and possessive forms of these abbreviations parallel the usual conventions. Thus r.vs. abbreviates "random variables" and i.c.s.'s means "internal control system's."

This appendix illustrates the chapter (or appendix), section and paragraph three level numbering system used in this dissertation. In order to avoid burdening the reader with superfluous numerical details, equations are numbered by section. Thus the equation

$$E_\beta(x|\,p,q) = \int_0^1 x f_\beta(x|\,p,q)\,dx \tag{1}$$

is referred to as (1) within this section. References from outside of section A1.2 refer to the equation as A1.2(1).

In contrast, a reference to paragraph 1 of section A1.2 is always referred to as paragraph A1.2.1. If paragraph 1 was part of chapter 1, it would be referred to as paragraph 1.2.1. Similarly, table 1.2.1 is the first table of section 1.2 in chapter 1. Thus, equations, paragraphs, tables and figures are all numbered by section. For equations the intrasection reference is shortened to (1).

A1.2.2 Mathematical Statistics Notation

The notation used in this dissertation is based upon the notation of Kendall and Stuart (1958) for probability moments and Raiffa and Schlaifer (1961) for p.d.fs. A composition of these two systems leads to the following conventions.

(i) The letter E is used to indicate the expected value operator. When it is useful to emphasize the nature of the p.d.f. and/or the distribution's r.v. subscripted forms such as E_β, $E_{\beta x}$ and E_x are used. Subscripts such as β, γ, and N are used to indicate a specific type of p.d.f.

(ii) The symbol | is used to initiate an optional list of parameters in an expression for the p.d.f. of a r.v. For example $f_\beta(x|p,q)$ and $E_\beta(x|p,q)$ indicate the p.d.f. and expected value of a standardized beta distribution with parameters p and q. The forms $f_\beta(x)$ and $E_\beta(x)$ are used when the nature of the parameter set is apparent from the supporting text.

(iii) The symbols μ_r and μ_r' are used to indicate, respectively, the r^{th} central moment and the r^{th} noncentral moment (about zero) of an implicit probability distribution. When an explicit statement of the probability distribution is useful, forms such as $\mu_r'(\beta|p,q)$ and $\mu_r(\beta)$ are used. In applications, where a number of different noncentral forms of a common p.d.f. f(x) are considered the noncentral moments are specified by expressions such as $\mu_r(\frac{x-a}{b-a})$ or $\mu_r(x)$. Thus μ_r', $\mu_r'(\beta)$ and $\mu_r(x)$ all indicate the $E_{\beta x}(x^r)$.

(iv) The letters s and e are used to emphasize, respectively, the standardized and extended forms of a probability distribution. Thus, $\mu_r'(s)$ and $E_s(x^r)$ indicate the r^{th} noncentral moment of a standardized probability distribution, while $f_{e\beta}(x)$, $\mu_r(e\beta)$ and $E_{e\beta}(u-\mu_1')^r$ refer to the p.d.f. and r^{th} central moment of an extended beta distribution. When this convention is used in conjunction with a specified r.v. a form such as $f_{e\beta x}(\frac{y-a}{b-a})$ could be used. The extended beta p.d.f., $f_{e\beta}(x)$, has been specified with a substitution defined by $x = \frac{y-a}{b-a}$.

A1.3 A Restatement of the Cushing Analysis in the Yu and Neter Format

The purpose of this section is to illustrate how the error analysis model for i.c.s. document processing developed by Cushing (1974) can be reformulated in the notation used by Yu and Neter (1973) for describing a document flow. Yu and Neter presented their model using four possible document states. However, as they note, their procedures are equally applicable for other state vectors. The present discussion is based upon a two state model: not in error (NE) and in error (E).

Cushing analyzed a number of scenarios, the simplest of which is a "Single Control--Single Error." Table A1.3.1 gives the steps used to restate this component of the Cushing system in the Yu and Neter format. More elaborate components of the Cushing analysis can be reformulated in the Yu and Neter format using similar procedures. Thus rather than representing two disjoint analyses of i.c.ss., the models of these authors are compatible systems that emphasize different aspects of the same subject.

A1.4 The Calculation of Joint Moments for Correlated Random Variables

This section discusses procedures that can be used to calculate joint probability moments when there is statistical dependence between several of the component r.vs. This problem arose in sections 5.3 and 6.3 where methods were developed for determining the moments of consolidated error rate r.vs.

A1.4.1 Some Possible Approaches to the Analysis

It is assumed that the decision maker's knowledge about the natural of the correlation between r.vs. is somewhat vague. Consequently, an analysis of the effect of correlation on the calculation of joint

Table A1.3.1

A Restatement of the Single Control--Single Error Component of the Cushing System in the Yu and Neter Framework

The Two States: Not in Error (NE), In Error (E)

The Initial State Probabilities: $W_I = [1,0]$

The "Processing" Transformation--Probability Matrix (P):

	NE	E
NE	p	1-p
E	0	1

The Output Vector: $W_o = W_I P = [p, 1-p]$

The Control Step Branching Operation:

Q": Error Processing

	NE	E
NE	1-P(s)	0
E	0	P(e)

Q': No Error Processing

	NE	E
NE	P(s)	0
E	0	1-P(e)

The After Branching State Output Vectors:

$W''_o = W_o Q'' = [p(1-P(s)),\ (1-p)P(e)]$ (error processing)

$W'_o = W_o Q' = [pP(s),\ (1-p)(1-P(e))]$ (no error processing)

The "Error Processing" Transformation Probability Matrix (R):

	NE	E
NE	P(d)	1-P(d)
E	P(c)	1-P(c)

The Rejoined Vector:

$$W'_o + W''_o R = [pP(s),(1-p)(1-P(e))] + [p(1-P(s)),(1-p)(P(e)]\begin{pmatrix} P(d) & 1-P(d) \\ P(c) & 1-P(c) \end{pmatrix}$$

$$= [pP(s) + p(1-P(s))P(d) + (1-p)P(e)P(c),$$

$$(1-p)(1-P(e)) + p(1-P(s))(1-P(d)) + (1-p)P(e)(1-P(c))]$$

(The Cushing Result)

moments should be adequately represented by an appropriately constructed multivariate probability model. The following two requirements for such a model would seem to be necessary. The marginal distribution for each variable of the multivariate model should match the known p.d.f. (or moments) of each component r.v. within the joint moments. And second, sufficient parameters should be available in the model for specifying an independent estimate of the usual correlation between each pair of r.vs.

If the known marginal distributions were normally distributed, a multivariate normal distribution would satisfy these requirements. However, the marginal distributions of interest in the current cirucm-stances tend to be right skewed and highly leptokurtic with the general shape of beta distributions. For these distributions a multivariate normal model offers only a very crude approximation.

As an alternative, multivariate beta distributions might be considered. These distributions are surveyed by Johnson and Kotz (1972, pp. 186, 231-238) and by Press (1972, pp. 133-138). Also of interest is the analysis of Tan (1969). Unfortunately the multivariate beta distributions that have been developed lack the parametric richness of the multivariate normal distribution. While each marginal distribution may be a beta distribution, only one independent parameter can be used to specify its general form. Further, the correlation between r.vs. can not be freely specified as with a multivariate normal distribution.

Similar difficulties arise if one attempts to approximate each marginal distribution with a gamma p.d.f. The Wishart distribution and several other multivariate distributions with marginal gamma p.d.fs.

are reviewed by Johnson and Kotz (1972, pp. 158-169, 216-230). In all cases these multivariate distributions lack sufficient parameterization for the present purposes. The Wishart distributions present additional difficulties since only the "diagonal" marginal distributions are gamma distributed (Press 1972, pp. 100-105).

These difficulties motivate the transformation approaches developed in this section. It is assumed that a transformation of each known marginal distribution is normally distributed. A multivariate normal distribution is then constructed from these transformed r.vs. The moments and correlations of the original r.vs. are used to determine the multivarite parameters. Now if a unique inverse to each transformation exists, it is possible to express the desired joint moments in terms of multivariate normal expected values (Jones and Miller 1966). These multiple integrals can in theory be used to calculate the approximation to the desired joint moments.

In practice these steps can lead to analytical problems. There are, however, several special features of the logarithmic transformation ($Z = \log X$) that can be used to develop a simple solution. In the subsequent paragraphs this transformation and a semitractable power transformation ($Z = X^{1/k}$) are examined in detail.

The logarithmic transformation gives exact results when the original untransformed marginal distributions are lognormal distributed. For other types of marginal distributions, the nature of the lognormal approximation can be seen from a (β_1,β_2) chart for the Pearson family of distributions (Johnson and Kotz 1970a, pp. 14, 18). The lognormal line is below the gamma line or boundary of the beta distribution's skewness/kurtosis region.

The power transformation has been used in an analysis of beta distributions by Cole (1975). He used this approach in developing a Bayesian reliability procedure for a U.S. Naval Reliability Guide Series (Bird Engineering 1974). Paragraph A1.4.3 uses the power transformation with marginal beta p.d.fs. defined on [0,1]. The analysis illustrates a general approach which can be adapted to a number of other marginal distributions and transformations.

As a third alternative, an approximation developed by Boyd (1971) for beta distribution can be used. He showed that the r.v. $Z = 2\text{Arcsin}\sqrt{X}$ is very closely approximated by a normal distribution with

$$E(z) = 2 \text{ Arcsin } \left(\frac{p+1/2}{p+q+1}\right)^{1/2} \qquad \text{Var}(z) = (p+q-1)^{-1}$$

where X is a beta r.v. with parameters p and q (equation A2.1(1) with a = 0, b = 1). The joint moments of the r.vs. $X_i = i = 1,\ldots n$ are thus approximated by

$$E\left(\prod_{i=1}^{n} x_i^{r_i}\right) \cong E_{mN}\left(\prod_{i=1}^{n} (\text{Sin}(z_i/2))^{2r_i}\right) \tag{1}$$

Since numerical n-dimensional integration must be used to evaluate (1) the details of this approach have not been developed.

It is assumed in the subsequent analysis that the decisionmaker can estimate the product moment correlation for each pair of r.vs. that can be formed from the joint moments to be evaluated. The analysis is based upon a discrete value for each of these parameters. Thus, if a p.d.f. was specified for a particular correlation it would be necessary to integrate over this density function with respect to conditional values of the joint moments. The details of these extensions are not considered.

A1.4.2 A Logarithmic Transformation

In using a logarithmic transformation each known marginal distribution is in effect approximated by a lognormal distribution with the same mean and variance as the target marginal distribution. Thus, letting Z_i, Y_i and X_i be r.vs. for a normal, lognormal and marginal distribution, it is assumed that $Y_i = X_i - \theta_i$ and $Z_i = \log(X_i - \theta_i)$. The parameter θ_i specifies the lower limit of nonzero probability mass.

It is also assumed that the required joint moments to be calculated are relative to $(\theta_1,\ldots,\theta_n)$, and consequently of the form $E(\prod_{i=1}^{n}(x_i-\theta_i)^{r_i})$ where $r_i \geq 0$ for $i = 1,\ldots,n$. It follows that

$$E\Big(\prod_{i=1}^{n}(x_i-\theta_i)^{r_i}\Big) = E\Big(\prod_{i=1}^{n}\exp(r_i \log y_i)\Big) = E\big(\exp(R'Z)\big) \tag{2}$$

where

$$R' = (r_1,\ldots,r_n)$$

$$Z' = (z_1,\ldots,z_n) = (\log y_1,\ldots, \log y_n)$$

Now assuming that the random vector Z' has a multivariate normal distribution with $U' = (\mu_1,\ldots,\mu_n)$ and $V = (\sigma_{ij}) = \big(\mathrm{Cov}(z_i,z_j)\big)$ (where $\sigma_{ii} = \sigma_i^2$) it follows from (2) that (Johnson and Kotz 1972, p. 20)

$$E\Big(\prod_{i=1}^{n}(x_i-\theta_i)^{r_i}\Big) = \exp(R'U + \tfrac{1}{2}R'VR) \tag{3}$$

Equation (3) gives the required joint moments in terms of the mean and covariance matrices of the multivariate normal distribution. The equation also can be used to determine the numerical values for the parameters of this multivariate normal distribution. In order to insure that each lognormal marginal distribution has the same mean and variance as the initial marginal distribution, (3) can be evaluated using a single nonzero element of $R = (r_1,\ldots,r_k)$.

Thus, for $1 \leq k \leq n$ letting $r_k = 1$ or 2 and $r_i = 0$ for $i \neq k$ it follows from (3) that

$$\mu_k + (1/2)\sigma_k^2 = \log[E(x_k-\theta_k)] \tag{4}$$

$$2\mu_k + 2\sigma_k^2 = \log[E(x_k-\theta_k)^2]$$

Given numerical values for the r.h.s. moments of (4), the required parameter values for μ_k and $\sigma_k^2 = \sigma_{kk}$ are readily determined.

Equation (3) also can be used to determine the nondiagonal elements of V given numerical values for the correlation between the r.vs. X_i and X_j. Letting $r_i = r_j = 1$ with all the other r's equal to zero yields

$$E\big((x_i-\theta_i)(x_j-\theta_j)\big) = \exp\big(\mu_i+\mu_j+(1/2)(\sigma_i^2+\sigma_j^2)+\sigma_{ij}\big) \tag{5}$$

The l.h.s. of (5) can be expressed in terms of the estimated correlation, $\rho(x_i,x_j)$ between x_i and x_j by observing that

$$\rho(x_i,x_j) = \frac{E(x_i x_j) - E(x_i)E(x_j)}{[\mathrm{Var}(x_i)\ \mathrm{Var}(x_j)]^{1/2}} = \frac{E\big((x_i-\theta_i)(x_j-\theta_j)\big)-E(x_i-\theta_i)E(x_j-\theta_j)}{[\mathrm{Var}(x_i)\ \mathrm{Var}(x_j)]^{1/2}} \tag{6}$$

Solving (6) for $E((x_i-\theta_i)(x_j-\theta_j))$ and substituting into (5) yields

$$\begin{aligned}\sigma_{ij} = \log\,&[\rho(x_i,x_j)\ [\mathrm{var}(x_i)\mathrm{var}(x_j)]^{1/2} + E(x_i-\theta_i)E(x_j-\theta_j)] \\ &- [\mu_i + \mu_j + (1/2)(\sigma_i^2+\sigma_j^2)]\end{aligned} \tag{7}$$

In using the lognormal correlation model the $E(x_i-\theta_i)$ and $\mathrm{Var}(x_i)$ $i = 1,\ldots, n$ are first determined for the original r.vs. Next equations (4) are first solved for μ_k,σ_k $k = 1,\ldots,n$. Now using these results and estimates of $\rho(x_i,x_j)$ equation (7) is used to determine the off diagonal terms of V. These parameters are now used in (3) to determine the required joint moments.

A1.4.3 A Power Function Transformation

In this paragraph a correlation model is developed for low error rate r.vs. X_i with leptokurtic and right skewed marginal p.d.fs. While these assumptions are often satisfied by beta p.d.fs., the analysis is not necessarily restricted to this distribution. However, the ease with which noninteger moments can be calculated for the beta distribution does make the procedure ideally suited for beta p.d.f.

For each r.v. X_i, an integer k_i for the transformation $Y_i = X^{1/k_i}$ is desired such that the r.v. Y_i is approximately normally distributed. The parameters k_i can be selected in many ways. Cole (1975), for example, assumed that a reliability r.v. $T_i = 1 - X_i$ had a beta p.d.f. and suggested that k_i be an integer value such that the $E(t_i^{k_i}) = 0.5$

In the current development it is assumed that k_i is selected to minimize the skewness of $Y_i = X_i^{1/k_i}$. Using a third moment measure of skewness it follows that the optimal k_i is a solution to

$$\mu_3^*(y_i) = \min_{k_i} \mu_3(y_i)$$

where

$$\begin{aligned}\mu_3(y_i) &= E\big(y_i - E(y_i)\big)^3 = E(y_i^3) - 3E(y_i^2)E(y_i) + 2E^3(y_i) \\ &= E(x_i^{3/k_i}) - 3E(x_i^{2/k_i})E(x_i^{1/k_i}) + 2E^3(x_i^{1/k_i})\end{aligned} \tag{8}$$

When X_i is beta distributed r.v. it follows from Johnson and Kotz (1970b, p. 40) that

$$E(x_i^{r/k_i}) = B(p+r/k_i, q)/B(p,q) \qquad r > 0 \tag{9}$$

Now setting $r = 1,2$, and 3 in (9) and using these results in conjunction with (8) it is possible to iteratively search for the optimal k_i. For instance, for $p = 3$ and $q = 101$ the integer $k = 3$ yields a very symmetric bell shaped p.d.f.

The optimal k_i can be used with equation (9) for $r = 1$ and 2 to find the mean and variance of the r.v. $Y_i = X_i^{1/k_i}$. Repeating this process for $i = 1,\ldots,n$ yields the mean value vector U' and diagonal elements of V, the covariance matrix of the approximating multivariate normal distribution.

Defining $b_i = 1/k_i$ and $b_j = 1/k_i$, the off diagonal covariance elements of V are given by

$$\mathrm{Cov}(y_i,y_j) = E(x_i^{b_i}x_j^{b_j}) - E(x_i^{b_i})E(x_j^{b_j}) \tag{10}$$

It is now assumed that $x_i^{b_i}\ x_j^{b_j}$ can be approximated by a two dimensional truncated Taylor series with quadratic terms (Buck 1956, p. 200). Expanding $x_i^{b_i}x_j^{b_j}$ about the mean values

$$m_i = E(x_i) \qquad m_j = E(x_j) \tag{11}$$

and then taking expected values yields

$$\begin{aligned} E(x_i^{b_i}x_j^{b_j}) \cong\ & m_i^{b_i}m_j^{b_j} + (1/2)(b_i)(b_i-1)m_i^{b_i-2}m_j^{b_j}\ \mathrm{Var}(x_i) \\ & + (1/2)(b_j)(b_j-1)m_i^{b_i}m_j^{b_j-2}\ \mathrm{Var}(x_j) \\ & + b_ib_jm_i^{b_i-1}m_j^{b_j-1}E((x_i-m_i)(x_j-m_j)) \end{aligned} \tag{12}$$

Using the estimated correlation $\rho(x_i,x_j)$ it follows that

$$E\big((x_i-m_i)(x_j-m_j)\big) = \rho(x_i,x_j)[\mathrm{Var}(x_i)\ \mathrm{Var}(x_j)]^{1/2} \tag{13}$$

The $E\big((x_i-m_i)(x_j-m_j)\big) = \rho(x_i,x_j)[\mathrm{Var}(x_i)\ \mathrm{Var}(x_j)]^{1/2}$ by substituting (11) and (13) into (12), and then in turn substituting (12) and (9) with r = 1 into (10).

Assuming that

$$E\Big(\prod_{i=1}^{n} x_i^{r_i}\Big) = E_{mN}\Big(\prod_{i=1}^{n} y_i^{k_i r_i}\Big) \tag{14}$$

it follows that the joint moments of the X_i r.vs. can be determined from the joint integer moments of a multivariate normal distribution.

It is usually suggested that such multivariate moments be determined by differentiating the multivariate characteristic function (Johnson and Kotz 1972, pp. 39-40)

$$\Phi(T) = \exp(T'U + \frac{1}{2} T'VT) \tag{15}$$

While this procedure is convenient for lower order joint moments, the integers k_i of (14) scale up the order of the required moments. A general expression based upon (15) for such higher order moments is neither apparent from the literature nor easily developed.

A more tractable procedure for evaluating (14) can be developed using equations determined by Bergström (1918). His formula for the joint moments of a standardized multivariate normal distribution can be adapted to computer processing. However, in order to use this procedure (14) must be written in standardized form. Letting $Z_i = (Y_i-\mu_i)/\sigma_i$ and defining $a_i = k_i r_i$, equation (14) becomes

$$E\left(\prod_{i=1}^{n} x_i^{r_i}\right) = E\left(\prod_{i=1}^{n} (\sigma_i z_i + \mu_i)^{a_i}\right)$$

$$= E\left(\prod_{i=1}^{n} \sum_{j_i=0}^{a_i} \binom{a_i}{j_i} \mu_i^{a_i - j_i} \sigma_i^{j_i} z_i^{j_i}\right)$$

$$= \sum_{j_1=0}^{a_1} \cdots \sum_{j_n=0}^{a_n} \left[\prod_{i=1}^{n} \binom{a_i}{j_i} \mu_i^{a_i - j_i} \sigma_i^{j_i}\right] E\left(\prod_{i=1}^{n} z_i^{j_i}\right) \tag{16}$$

Bergström's procedure can now be used to evaluate each expected value in the r.h.s. of (16). His results are given in terms of the correlation matrix of the random vector $Z = (Z_1, \ldots, Z_n)$ defined by

$$R = (r_{ij}) = \left(\mathrm{Cov}(y_i, y_j)/\sigma_i \sigma_j\right) \tag{17}$$

Bergström showed that* the

$$E\left(\prod_{i=1}^{n} z_i^{a_i}\right) = \sum_{S} \prod_{i=1}^{n} \left[\frac{a_i!}{(2e_{ii})!!} \left(\prod_{j>i}^{j=n} \frac{r_{ij}^{e_{ij}}}{e_{ij}!}\right)\right] \tag{18}$$

where S is the set of all nonnegative integer valued solutions, $S = (e_{ij}; o \underline{<} i \underline{<} j \underline{<} n)$, to the system of equations

$$\begin{aligned} 2e_{11} + e_{12} + \ldots + e_{1n} &= a_1 \\ e_{12} + 2e_{22} + \ldots + e_{2n} &= a_2 \\ &\vdots \\ e_{1n} + e_{2n} + \ldots + 2e_{nn} &= a_n \end{aligned} \tag{19}$$

and $(2e_{ii})!! = (2e_{ii})(2e_{ii}-2)\ldots 4.2$

As an example, for $n = 2$ with $a_1 = 4$ and $a_2 = 2$ equations (19) have solutions $s_1 = (e_{11}=1,\ e_{22}=0,\ e_{12}=2)$ and $s_2 = (e_{11}=2,\ e_{22}=1,\ e_{12}=0)$ Now from (18) it follows that

$$E(z_1^4 z_2^2) = \frac{4!}{2!!}\frac{r_{12}^2}{2!}\frac{2!}{0!} + \frac{4!}{4!!} = 12r_{12}^2 + 3$$

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*Bergström's notation has been changed slightly to clear up some ambiguities, correct a topographical error and put (19) into a more convenient form for computer evaluation. Berström also calculated a number

The set S of solutions to (19) can be determined using a computer search procedure. Table A1.4.1 illustrates for $n = 4$ how a set of nested "DO LOOPS" can be used to find the elements of S. Similar procedures can be established for other values of n. Rather than writing exact coding the table outlines the general approach of such coding using more convenient notation.

Table A1.4.1

A Computer Procedure for Evaluating the Equations A1.4(19)

DO 10 $e_{11} = 0, I(a_1/2)$

DO 10 $e_{22} = 0, I(a_2/2)$

DO 10 $e_{33} = 0, I(a_3/2)$

DO 10 $e_{44} = 0, I(a_4/2)$

DO 10 $e_{12} = 0, \min\,(a_1-2e_{11}, a_2-2e_{22})$

DO 10 $e_{13} = 0, \min\,(a_1-2e_{22}-e_{12}, a_3-2e_{33})$

DO 10 $e_{23} = 0, \min\,(a_2-2e_{22}-e_{12}, a_3-2e_{33}-e_{13})$

$e_{14} = a_1 - 2e_{11} - e_{12} - e_{13}$

$e_{24} = a_2 - 2e_{22} - e_{12} - e_{23}$

$e_{34} = a_3 - 2e_{33} - e_{13} - e_{23}$

IF $(e_{14} + e_{24} + e_{34} + 2e_{44} = a_4)$ Store the e's.

10 CONTINUE

NOTE: $I(a_i/2)$ is the truncated integer quotient of $a_i/2$.

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of lower order moments using his method. These results correspond to similar calculations made by Wicksell (1919) using a characteristic function approach.

APPENDIX 2

MOMENT, CUMULANT AND OTHER PROPERTIES OF

BETA DISTRIBUTIONS

A2.1 Preface

This appendix brings together in a single source a number of distributional properties useful when working with beta distributions. Several general properties of moments useful in similar circumstances are also discussed.

In this dissertation the p.d.f.

$$f_\beta(t|p,q,a,b) = \frac{(t-a)^{p-1}(b-t)^{q-1}}{B(p,q)(b-a)^{p+q-1}} \qquad \begin{array}{l} a \leq t \leq b \\ p > o \quad q > o \\ n = p + q > o \end{array} \tag{1}$$

is referred to as an "extended" beta distribution in contrast to the "standardized" beta distribution defined on $[0,1]$ with $a = 0$ and $b = 1$ and denoted by $f_\beta(t|p,q)$. When there is no possibility of confusion the adjective "standardized" is not used when referring to the latter distribution. The adjective "extended" has been adopted to avoid confusion with the term "generalized" beta distribution which has recently been used by Kattakkuzhy (1975) in another context.

In section A2.2 the noncentral and central moments for (1) are derived. While these are routine calculations, they are not available for reference in mathematical journals, or in standard sources such as Johnson and Kotz (1970), Raiffa and Schlaifer (1961) or, for example, in a recent collection of properties of distributions by Hastings and Peacock (1974).

Section A2.3 develops formulas for calculating the parameters of (1) using the moments of a target distribution to be approximated by (1). Special cases of the approximation with preset values for the parameters a and b, or only the parameter a, are also considered.

Section A2.4 derives recursive formulas for the cumulants and moments of (1). The cumulant derivations are based upon the work of Breitenberger (1959). The analysis adds some mathematical details and rigor to the Breitenberger development. However, the basic result remains as originally stated. Breitenberger's contribution is embedded in an obscure government technical memorandum and has apparently never been recognized by authors such as Johnson and Kotz or published in a mathematical journal.*

Section A2.5 gives a number of recently published miscellaneous results useful in error rate analyses. Besides presenting each result the discussion focuses on a few related issues not considered by the originating author.

The results of this appendix are used in this dissertation to proceed from beta distributions to their moments, or vice versa; to develop a moment approximation to the beta-normal distribution; and to approximate a sum of beta-normal r.vs. This latter task is conveniently developed in terms of cumulants since the cumulants of a sum of independent random variables is just the sum of the cumulants of each

*An extensive search using reference sources such as the Information Access Series (1973-1975), Science Citations (1967-1976), Mathematical Review (1940-1972), etc., failed to uncover any references to Breitenberger's work or equivalent analysis.

variable (see Johnson and Kotz 1969, pp. 20-21). This property of cumulants has motivated the dissertation's interest in cumulants as discussed in section A2.3. Formulas for converting cumulants to moments and vice versa are given by Kendall and Stuart (1958, pp. 68-71) and are not repeated here.

A2.2 The Probability Moments of the Extended Beta Distribution

Moment equations for the extended beta distribuiton can be derived from the noncentral and central moments of the standardized beta distribution. In this section these standardized results are presented and then used to develop the moments of the extended beta distribution.

A2.2.1 The Noncentral and Central Moments of the Standardized Beta Distribution

The standardized form of the beta distribution is given by setting $a = 0$ and $b = 1$ in A2.1(1). The noncentral moments can be easily derived by direct integration. This yields (Johnson and Kotz 1970b, p. 40)

$$\mu'_r(s) = \frac{\Gamma(p+r)\Gamma(p+q)}{\Gamma(p)\Gamma(p+q+r)}$$

For integer $r > 0$, with $n = p + q$ this reduces to

$$\mu'_r(s) = \mu'_{r-1}(s)\left(\frac{p+r-1}{n+r-1}\right) = \prod_{i=1}^{r}\left(\frac{p+i-1}{n+i-1}\right) \tag{1}$$

The recursive form of (1) is particularly useful when higher moments are required. For $r = 1,2,3,4$ the nonrecursive form of equation (1) becomes

$$\mu_1'(s) = \frac{p}{n} \qquad \mu_2'(s) = \frac{p(p+1)}{n(n+1)}$$

$$\mu_3'(s) = \frac{p(p+1)(p+2)}{n(n+1)(n+2)} \qquad \mu_4'(s) = \frac{p(p+1)(p+2)(p+3)}{n(n+1)(n+2)(n+3)} \tag{2}$$

Equations for central moments in terms of noncentral moments follow immediately from the expected value definition. For $r > 0$ and $\mu_o'(s) = 1$ it follows that

$$\mu_r(s) = E_{s\beta}((t-\mu_1'(s))^r|p,q) = E_{s\beta}\left(\sum_{k=o}^{r}(-1)^k\binom{r}{k}t^{r-k}\mu_1'(s)^k\right)$$

$$= \sum_{k=o}^{r}(-1)^k\binom{r}{k}\mu_1'(s)^k\mu_{r-k}'(s) \tag{3}$$

For $r = 1,2,3,4$ equation (3) can be evaluated using the noncentral moments of (2). Omitting some tedious algebraic manipulations, this yields

$$\mu_1(s) = 0 \tag{4}$$

$$\mu_2(s) = \mu_2'(s) - \mu_1'(s)^2 = \frac{p(p+1)}{n(n+1)} - \frac{p^2}{n^2} = \frac{p(n-p)}{n^2(n+1)}$$

$$\mu_3(s) = \mu_3'(s) - 3\mu_1'(s)\mu_2'(s) + 2\mu_1'(s)^3$$

$$= \frac{p(p+1)(p+2)}{n(n+1)(n+2)} - 3\,\frac{p^2(p+1)}{n^2(n+1} + 2\,\frac{p^3}{n^3} = \frac{2p(n-p)(n-2p)}{n^3(n+1)(n+2)}$$

$$\mu_4(s) = \mu_4'(s) - 4\mu_1'(s)\mu_3'(s) + 6\mu_1'(s)^2\mu_2'(s) - 3\mu_1'(s)^4$$

$$= \frac{p(p+1)(p+2)(p+3)}{n(n+1)(n+2)(n+3)} - \frac{4p^2(p+1)(p+2)}{n^2(n+1)(n+2)} + 6\,\frac{p^3(p+1)}{n^3(n+1)} - 3\,\frac{p^4}{n^4}$$

$$= \frac{3p(n-p)[2n^2+p(n-p)(n-6)]}{n^4(n+1)(n+2)(n+3)}$$

A2.2.2 The Noncentral Moments of the Extended Beta Distribution

The moments, $\mu_r'(e)$ for integer $r > 0$ are derived in terms of the noncentral standardized moments as follows.

$$\mu_r'(e) = E_{e\beta}(t^r|p,q,a,b) = (b-a)^r E_{e\beta}((\frac{t-a}{b-a} + \frac{a}{b-a})^r)$$

Now making the change of variable $y = (t-a)/(b-a)$ and expanding the resulting expression (with $\mu_0'(s)=1$) it follows that

$$\mu_r'(e) = (b-a)^r E_{s\beta}(\sum_{k=o}^{r} \binom{r}{k} y^{r-k}(\frac{a}{b-a})^k|p,q)$$

$$= \sum_{k=o}^{r} \binom{r}{k} a^k (b-a)^{r-k} \mu_{r-k}'(s) \tag{5}$$

For $r = 1,2,3,4$ equation (5) yields

$$\mu_1'(e) = (b-a)\mu_1'(s) + a = (b-a)\frac{p}{n} + a$$

$$\mu_2'(e) = (b-a)^2\mu_2'(s) + 2a(b-a)\mu_1'(s) + a^2 \tag{6}$$

$$= (b-a)^2 \frac{p(p+1)}{n(n+1)} + 2a(b-a)\frac{p}{n} + a^2$$

$$\mu_3'(e) = (b-a)^3\mu_3'(s) + 3a(b-a)^2\mu_2'(s) + 3a^2(b-a)\mu_1'(s) + a^3$$

$$= (b-a)^3 \frac{p(p+1)(p+2)}{n(n+1)(n+2)} + 3a(b-a)^2 \frac{p(p+1)}{n(n+1)} + 3a^2(b-a)\frac{p}{n} + a^3$$

$$\mu_4'(e) = (b-a)^4\mu_4'(s) + 4a(b-a)^3\mu_3'(s) + 6a^2(b-a)^2\mu_2'(s)$$
$$+ 4a^3(b-a)\mu_1'(s) + a^4$$

$$= (b-a)^4 \frac{p(p+1)(p+2)(p+3)}{n(n+1)(n+2)(n+3)} + 3a(b-a)^2 \frac{p(p+1)(p+2)}{n(n+1)(n+2)}$$

$$+ 6a^2(b-a)^2 \frac{p(p+1)}{n(n+1)} + 4a^3(b-a)\frac{p}{n} + a^4$$

A2.2.3 The Central Moments of the Extended Beta Distribution

The moments, $\mu_r(e)$ for integer $r > 0$ are derived in terms of standardized central moments using the expected value definition

$$\mu_r(e) = E_{e\beta}\ ((t-\mu_1'(e))^r | p,q,a,b)$$

Substituting for $\mu_1'(e)$ from (6) and making the change of variable $y = (t-a)/b-a$ it follows that

$$\mu_r(e) = E_{e\beta}((t-(b-a)\mu_1'(s)-a)^r) = (b-a)^r E_{e\beta}\ ((\frac{t-a}{b-a} - \mu_1'(s))^r)$$

$$= (b-a)^r E_{s\beta}\ ((y-\mu_1'(s))^r | p,q) = (b-a)^r \mu_r(s) \tag{7}$$

Replacing $\mu_r(s)$ of (7) by (3) gives the alternative expression

$$\mu_r(e) = (b-a)^r \sum_{k=o}^{r} (-1)^k \binom{r}{k} \mu_1'(s)^k \mu_{r-k}'(s) \tag{8}$$

Using the equations for $\mu_r(s)$ given by (4), equation (7) yields for $r = 1,2,3,4$

$$\mu_1(e) = 0 \tag{9}$$

$$\mu_2(e) = (b-a)^2(\mu_2'(s)-\mu_1'(s)) = (b-a)^2 \frac{p(n-p)}{n^2(n+1)}$$

$$\mu_3(e) = (b-a)^3(\mu_3'(s)-3\mu_1'(s)\mu_2'(s)+2\mu_1'(s)^3)$$

$$= (b-a)^3 \frac{2p(n-p)(n-2p)}{n^3(n+1)(n+2)}$$

$$\mu_4(e) = (b-a)^4(\mu_4'(s)-4\mu_1'(s)\mu_3'(s)+6\mu_1'(s)^2\mu_2'(s)-3\mu_1'(s)^4)$$

$$= (b-a)^4 \frac{3p(n-p)[2n^2+p(n-p)(n-6)]}{n^4(n+1)(n+2)(n+3)}$$

A2.3 Using Moments to Fit an Extended Beta Distribution

This section develops procedures for fitting an extended beta distribution to a target r.v. $\tilde{X}$ with known central or noncentral moments The objective of this section is thus just the opposite of section A2.2

where known parameters of an extended beta distribution are used to determine the moments. The analysis of the section is developed in terms of the central moments of the target r.v. to be approximated by an extended beta distribution. When the noncentral moments are initially available, the required central moments must be first calculated using

$$\mu_2 = E(x-\mu_1')^2 = \mu_2' - (\mu_1')^2 \tag{1}$$

$$\mu_3 = E(x-\mu_1')^3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$\mu_4 = E(x-\mu_1')^4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

<u>A2.3.1 A Solution System</u>

The analysis follows a rather standard procedure. The moments of the extended beta distribution are expressed in terms of the distribution's parameters. The target r.v.'s known moments (i.e., numerical values) are substituted for the beta moments and the resulting system of equations is then solved for the beta parameters.

One possible solution system can be immediately written down using the equation A2.2(6i) for $\mu_1'(e)$ and equations A2.2(9). However, it is possible to develop an alternative solution system using A2.2(1). This approach is particularly useful when the parameter a or the parameters a and b are already known. It follows from A2.2(1) for integer $r > 0$ that

$$\prod_{i=1}^{r} \left(\frac{p+i-1}{n+i-1}\right) = \mu_r'(s) = E_{s\beta}(x^r|p,q) \tag{2}$$

Now making the change of variable $x = (t-a)/(b-a)$, (2) can be written as

$$\prod_{i=1}^{r} \left(\frac{p+i-1}{n+i-1}\right) = E_{e\beta}\left(\left(\frac{t-a}{b-a}\right)^r \middle| p,q,a,b\right) \tag{3}$$

Substituting $(t-a)^r = [(t-\mu_1'(e)) + (\mu_1'(e)-a)]^r$ into (3) and letting $\mu_0(e) = 1$, $\mu_1(e) = 0$ it follows that

$$(b-a)^r \prod_{i=1}^{r} \left(\frac{p+i-1}{n+i-1}\right) = \sum_{k=0}^{r} \binom{r}{k} \mu_{r-k}(e)(\mu_1'(e)-a)^k \tag{4}$$

On substituting into (4) μ_1', μ_2, μ_3 and μ_4 of the target r.v. it follows that

$$(b-a)\,\frac{p}{n} = \mu_1' - a = k_1 \tag{5}$$

$$(b-a)^2 \frac{p(p+1)}{n(n+1)} = \mu_2 + (\mu_1'-a)^2 = k_2 \tag{6}$$

$$(b-a)^3 \frac{p(p+1)(p+2)}{n(n+1)(n+2)} = \mu_3 + 3(\mu_1'-a)\mu_2 + (\mu_1'-a)^3 = k_3 \tag{7}$$

$$(b-a)^4 \frac{p(p+1)(p+2)(p+3)}{n(n+1)(n+2)(n+3)} = \mu_4 + 4(\mu_1'-a)\mu_3 + 6(\mu_1'-a)^2\mu_2' + (\mu_1'-a)^4 = k_4 \tag{8}$$

where for subsequent reference the constants k_1, k_2, k_3 and k_4 are defined as indicated.

A2.3.2 The Solution When the Constant Constant a is Known

Since the higher moments of a distribution tend to weight heavily the extremely right tail of a highly skewed r.v. it is advisable under these circumstances to stabilize the left side of a beta approximation by prespecifying the parameter a.* If this is possible, the remaining three parameters of the extended beta distribution are found by solving (5), (6) and (7).

---

*Pearson (1963) has examined the influence of tail probabilities on higher moments. Patnaik (1949) and Pearson (1959) have briefly considered similar issues in approximating a noncentral chi-squared distribution by a chi-squared distribution.

From (5) and (6) it follows that $k_2/k_1^2 = n(p+1)/p(n+1)$ or

$$n(p+1-c_1p) = c_1p \tag{9}$$

where $c_1 = k_2/k_1^2$

Similarly from (5), (6) and (7) it follows that $k_3/k_1k_2 = n(p+2)/p(n+2)$ or

$$n(p+2-c_2p) = 2c_2p \tag{10}$$

where $c_2 = k_3/k_1k_2$

Combining (9) and (10) yields

$$p = \frac{2(c_1-c_2)}{2c_2 - c_1c_2 - c_1} \tag{11}$$

The solution sequence is p, n, q = n-p and then b. Using the moments of the target r.v. k_1, k_2 and k_3 are calculated. These are in turn used to calculate c_1, c_2 and then p, from which (9) or (10) can be used to calculate n. Finally b is determined from (5).

A2.3.3 The Solution When Both a and b are Known

Johnson and Kotz (1970b, p. 44) derive equations for p and q when both a and b are known. Using (5) and (6) a computationally simpler alternative procedure can be derived. From these equations it follows that

$$\frac{p+1}{n+1} = \frac{k_2/k_1}{b-a} \qquad p = \frac{k_1}{b-a}\,n \tag{12}$$

Through simple algebra (12) can be solved for

$$n = \frac{k_2-k_1(b-a)}{k_1^2-k_2} \tag{13}$$

The solution sequence is to determine k_1, k_2 from (5) and (6). Using (13) n is calculated. This is in turn used in (12) to calculate p. Finally, q = n - p.

A2.3.4 The Solution When Both a and b are Unknown

For the general case when both a and b are unknown, a procedure paralleling the analysis of Elderton and Johnson (1969, pp. 57-58) can be used to solve equations A2.2(9) for (b-a), p, q. Equation A2.2(6i) is then used to find a and b.

Johnson and Kotz (1970a, pp. 4, 44) also give (without proof) a solution based upon the Elderton and Johnson procedure. However, their solution implicitly assumes that the mode of the target r.v. is also known. Since the transition from the Elderton and Johnson notation is rather cumbersome, the desired solution is now derived using their general technique.

Letting I = (b-a), n = p + q, E = pq = p(n-p) equations A2.2(9) are

$$\mu_2 = \frac{I^2E}{n^2(n+1)} \qquad \mu_3 = \frac{2I^3E(n^2-4E)^{1/2}}{n^3(n+1)(n+2)} \tag{14}$$

$$\mu_4 = \frac{3I^4E[E(n-6)+2n^2]}{n^4(n+1)(n+2)(n+3)}$$

In terms of the squared skewness and the kurtosis equations (14) become

$$\beta_1 = \mu_3^2/\mu_2^3 = \frac{4(n^2-4E)(n+1)}{E(n+2)^2} \quad \text{or} \quad \frac{\beta_1(n+2)^2}{4(n+1)} = \frac{n^2}{E} - 4 \tag{15}$$

$$\beta_2 = \mu_4/\mu_2^2 = \frac{3(n+1)[2n^2+E(n-6)]}{E(n+2)(n+3)} \quad \text{or} \quad \frac{\beta_2(n+2)(n+3)}{3(n+1)} = \frac{2n^2}{E} + (n-6) \tag{16}$$

and hence on subtracting (16) from (15)

$$\frac{\beta_1(n+2)^2}{2(n+1)} - \frac{\beta_2(n+2)(n+3)}{3(n+1)} = -8 - n + 6 = -(n+2) \tag{17}$$

Multiplying (17) by $6(n+1)/(n+2)$ and solving for n yields

$$n = \frac{6(\beta_2-\beta_1-6)}{3\beta_1-2\beta_2+6} \tag{18}$$

The solution for n given by (18) can be used in (15) to find

$$E = \frac{n^2}{4+(1/4)\beta_1\frac{(n+2)^2}{(n+1)}} \tag{19}$$

The solutions for n and E are then used in (14) to find

$$I = n\left(\frac{(\mu_2)(n+1)}{E}\right)^{1/2} \tag{20}$$

From the definitions $pq = E$ and $p + q = n$ it follows that $p^2 - pn + E = 0$, and hence

$$p = \frac{1}{2}[n \pm (n^2-4E)^{1/2}] \quad \text{and} \quad q = n - p \tag{21}$$

where the appropriate root is selected so $p > 0$, $q > 0$ and $p < q$ if $\mu_3 > 0$. From A2.2(6i) it follows that

$$a = \mu_1' - I(p/n) \qquad b = I - a \tag{22}$$

The solution sequence is thus to first calculate β_1, β_2 from μ_2, μ_3 and μ_4; and then progressively calculate (18) through (22).

A2.4 Recursive Cumulant and Moment Procedures for Beta Distrbutions

The first part of this section presents a cumulant recursive procedure for the extended beta distribution. A moment recursive procedure for the standardized beta distribution is then presented.

This latter result is also applicable to other distributions of the Pearson system of distributions. It is generally advisable to utilize these procedures when computing a large number of cumulants or moments. This is especially true in a repetitive sensitivity analysis where the increased calculating efficiency may become significant.

A2.4.1 The Derivation of a Cumulant Recursive Relationship

Recursive formulas (14) through (18) below for the cumulants of the beta distribution (1) are derived in this section. The general approach of the analysis is due to Breitenberger (1959). Because of the obscurity of the source and general lack of recognition of the procedure, a complete derivation has been developed from Breitenberger's brief analysis.

$$f(t) = \frac{(t-a)^{\alpha-1}\,(b-t)^{\beta-1}}{B(\alpha,\beta)\,(b-a)^{\alpha+\beta-1}} \qquad \alpha > 0 \qquad \beta > 0 \tag{1}$$

From (1) it follows that

$$f'(t) = \frac{(t-a)^{\alpha-2}(b-t)^{\beta-2}}{B(\alpha,\beta)\,(b-a)^{\alpha+\beta-1}}\,[(\alpha-1)(b-t)-(\beta-1)(t-a)] \tag{2}$$

Letting

$$\tau = (\alpha-1)b + (\beta-1)a \tag{3}$$

equation (2) can be rewritten using (1) as

$$f'(t) = \frac{f(t)[\tau-(\alpha+\beta-2)t]}{(t-a)(b-t)} \tag{4}$$

Multiplying the denominator out and writing (4) in differential equation form yields

$$-t^2f' + (a+b)tf' - abf' + (\alpha+\beta-2)tf - \tau f = 0 \tag{5}$$

On multiplying (5) by e^{st} and integrating from a to b, (5) can be written in the integral transformation form as

$$-M_s(t^2f') + (a+b)M_s(tf') - abM_s(f') + (\alpha+\beta-2)M_s(tf) - \tau M_s(f) = 0 \tag{6}$$

where

$$M_s(g) = \int_a^b e^{st}g(t)dt$$

Thus, for the p.d.f. f(t) defined by (1) it follows that

$$M_s(f) = \int_a^b e^{st}f(t)dt \tag{7}$$

is the usual moment generating function.

The general approach now to be taken is to convert (6) into a differential equation with respect to the cumulant generating function, and solve for recursive relationships between cumulants. The required substitution formulas are obtained by differentiating (7), and in a separate derivation by integrating (7) by parts followed by subsequent differentiation. This leads to*

$$\frac{d}{ds}M_s(f) = \int_a^b e^{st}[tf(t)]\,dt = M_s(tf) \tag{8}$$

---

*These steps are only valid when $\alpha > 1$ and $\beta > 1$ and hence $f(a) = f(b) = 0$. However, it will be seen that the results are unaffected by this restriction.

$$- sM_s(f) = \int_a^b e^{st}[f'(t)]dt = M_s(f') \tag{9}$$

$$\frac{d}{ds}(-sM_s(f)) = \int_a^b e^{st}[tf'(t)]dt = M_s(tf') \tag{10}$$

$$\frac{d^2}{ds^2}(-sM_s(f)) = \int_a^b e^{st}[t^2f'(t)]dt = M_s(t^2f') \tag{11}$$

Letting $M = M_s(f)$ it follows from (8) through (11) that (6) can be expressed as

$$(\frac{d}{ds})^2(sM) - (a+b)\frac{d}{ds}(sM) + absM + (\alpha+\beta-2)\frac{d}{ds}M - \tau M = 0 \tag{12}$$

Now let $\kappa_s(f) = \log M_s(f)$. On substituting the equivalent form, $M_s(f) = e^{\kappa_s(f)}$ into (12) and performing the indicated differenatiation it follows on cancelling out the common factor $e^{\kappa_s(t)}$ that (12) reduces to a differential equation in terms of the cumulant generating function

$$s\,[\kappa'' + (\kappa')^2] + [\alpha+\beta-(a+b)s]\kappa' + abs - [(a+b) + \tau] = 0 \tag{13}$$

Now one solution of (13) is given by $\kappa_s(f) = \log M_{s(f)}$. Expanding the cumulant generating function $\kappa_s(f)$ into a Taylor series leads to

$$\kappa(s) = \kappa_1 s + \frac{\kappa_2 s^2}{2!} + \frac{\kappa_3}{3!}s^3 + \ldots \tag{14}$$

$$\kappa'(s) = \kappa_1 + \kappa_2 s + \frac{\kappa_3}{2!}s^2$$

$$\kappa''(s) = \kappa_2 + \frac{2\kappa_3}{2!}s + \frac{3\kappa_4}{3!}s^2$$

where by definition κ_i is the i^{th} cumulant of the beta r.v.

Equations (14) are now substituted into (13). By collecting terms for each power of s the desired recursive relationships can be found.

A2.4.2 The Recursive Relationship for Beta Cumulants

For s^o the terms are

$$(\alpha+\beta)\kappa_1 - [a+b+\gamma] = 0$$

Using (3) this reduces to

$$(\alpha+\beta)\kappa_1 = \alpha b+\beta a \tag{15}$$

For s^1

$$\kappa_2 + \kappa_1^2 + (\alpha+\beta)\frac{\kappa_2}{1!} - (a+b)\kappa_1 + ab = 0$$

$$(\alpha+\beta+1)\kappa_2 = (a+b)\kappa_1 - \kappa_1^2 - ab \tag{16}$$

For s^2

$$\frac{2}{2!}\kappa_3 + (\kappa_1\frac{\kappa_2}{1!} + \frac{\kappa_2}{1!}\kappa_1) + (\alpha+\beta)\frac{\kappa_3}{2!} - (a+b)\frac{\kappa_2}{1!} = 0$$

$$(\alpha+\beta+2)\kappa_3 = 2(a+b)\kappa_2 - 2[\binom{1}{0}\kappa_1\kappa_2 + \binom{1}{1}\kappa_2\kappa_1]$$

$$= 2(a+b)\kappa_2 - 4\kappa_1\kappa_2 \tag{17}$$

For s^3

$$\frac{3\kappa_4}{3!} + [\kappa_1\frac{\kappa_3}{2!} + \frac{\kappa_2}{1!}\frac{\kappa_2}{1!} + \frac{\kappa_3}{2!}\kappa_1] + (\alpha+\beta)\frac{\kappa_4}{3!} - (a+b)\frac{\kappa_3}{2!} = 0$$

$$\begin{aligned}(\alpha+\beta+3)\kappa_4 &= 3(a+b)\kappa_3 - 3[\binom{2}{0}\kappa_1\kappa_3 + \binom{2}{1}\kappa_2\kappa_2 + \binom{2}{2}\kappa_3\kappa_1] \\ &= 3(a+b)\kappa_3 - 6(\kappa_1\kappa_3+\kappa_2\kappa_2) \qquad (18)\end{aligned}$$

In general, for s^n

$$(\alpha+\beta+n)\kappa_{n+1} = n(a+b)\kappa_n - n\sum_{i=1}^{n}\binom{n-1}{i-1}\kappa_i\kappa_{n+1-i} \qquad (19)$$

Thus, with equations (15) through (19) the mean (κ_1), variance (κ_2) and higher order cumulants can be determined for a beta r.v. Note that by using standard equations for converting cumulants into moments (Kendall and Stuart 1958, pp. 68-70), (15) through (19) can be converted into recursive relationships for the beta p.d.f. moments.

Even though the derivation of (15) through (19) is only valid for $\alpha > 1$, $\beta > 1$, these equations are just algebraic identities valid for all possible parameter values. As a check on these results it can be shown that the first four cumulant relationships yield equations A2.2(9) for the central moments of an extended beta distribution. Also, as observed by Breitenberger, since (4) is the general form of the Pearson system of 1st degree differential equations (see Elderton and Johnson 1969, p. 35) the above method could be used to calculate recursive cumulant equations for any p.d.f. in the Pearson system.

A2.4.3 Moment Recursive Relationships for Beta Distributions

The following result given by Bowman and Shenton (1973) without proof or reference gives higher central moments of any distribution of the Pearson system in terms of lower central moments.

$$\nu_{n+1} = \frac{n}{\Delta n}\left[(\beta_2+3)\beta_1^{1/2}\nu_n + (4\beta_2-3\beta_1)\nu_{n-1}\right] \tag{20}$$

where in terms of the notation for a standardized beta distribution

$$\nu_0 = 1,\ \nu_1 = 0,\ \nu_n = \frac{\mu_n(s)}{\mu_2(s)} \tag{21}$$

$$\Delta_n = 6\beta_2 - 6\beta_1 - 6 - n(2\beta_2-3\beta_1-6)$$

$$\beta_1 = \frac{\mu_3(s)^2}{\mu_2(s)^3} \qquad \beta_2 = \frac{\mu_4(s)}{\mu_2(s)^2}$$

The following general recursive relationships between cumulants and moments are also given by Bowman and Shenton.

$$\sum_{i=0}^{n-1}\binom{n-1}{i}\kappa'_{n-i}\nu_i = \nu_n \qquad \nu_0 = 1,\ \nu_1 = 1 \tag{22}$$

$$\sum_{i=0}^{n-1}\binom{n-1}{i}\kappa_{n-i}\mu_i = \mu_n \qquad \mu_0 = 1,\ \mu_1 = 0 \tag{23}$$

$$\sum_{i=0}^{n-1}\binom{n-1}{i}\kappa_{n-i}\mu'_i = \mu'_n \qquad \mu'_0 = 1 \tag{24}$$

where ν_i is defined by (21) and $\kappa'_i = \frac{\kappa_i}{\sigma^i}$ is a standardized cumulant.

Equations (20) and (22) can be used together to recursively calculate the cumulants of any distribution in the Pearson system. For standardized beta distributions this is an alternative approach to using the extended beta recursive formula (19) with a = 0, b = 1. Note also that (24) and (19) can be used recursively to calculate the noncentral moments of the extended beta distribution. However, when a = 0, b = 1, the simple recursive procedure given by A2.2(1) should be used.

A2.5 Useful Procedures for Error Rate Analysis

This section gives a number of recently published results that can be useful in an error rate analysis. A procedure is presented for fitting a beta distribution using a specified mode and variance, for comparing two alternative prior beta p.d.fs. and for determining tail probabilities of a beta distribution. The literature of the product of beta distribution is discussed, a procedure is given for conducting a prior to posterior analysis in terms of prior and posterior moments, and finally a scaling result for extended beta distributions is derived.

A2.5.1 Fitting a Beta Distribution Using the Mode and Variance

The moment methods of section A2.3 for fitting an extended beta distribution are useful when there is an empirical or theoretical basis for developing the required moments. In specifying a prior judgment more intuitive inputs are usually required. One approach is to specify an upper and lower boundary, a most likely value and a measure of confidence.

In terms of an extended beta distribution these inputs correspond to the domain parameters [a,b], the mode and the standard deviation (σ). The mathematics of this approach are now summarized. The algebraic steps leading up to (5) below can be found in Jacobs (1971). The subsequent details are new developments.

The mode of the extended beta distribtuion A2.1(1) with $p > 1$, $q > 1$ is given by (Johnson and Kotz 1970b, p. 41)

$$\text{Mode}(e\beta) = a + (b-a)\text{Mode}(s\beta) = a + (b-a)\frac{p-1}{p+q-2} \tag{1}$$

Defining

$$V = \frac{\sigma^2}{(b-a)^2} \qquad M = \frac{\text{Mode}(e\beta)-a}{b-a} \tag{2}$$

$$A = p - 1 \qquad B = q - 1 \tag{3}$$

equation (1) and equation A2.2(9ii) can be written as

$$B = \frac{A-MA}{M} \qquad V = \frac{p(n-p)}{n^2(n+1)} = \frac{(A+1)((B+1)}{(A+B+2)^2(A+B+3)} \tag{4}$$

The two equations in two unknowns given by (4) reduce to the cubic equation

$$(V)A^3 + (7VM-M^2+M^3)A^2 + (16VM^2-M^2)A + (12VM^3-M^3) = 0 \tag{5}$$

Thus, given a mode and variance, M and V can be calculated from (2). The cubic equation (5) is then solved for A. The results are used in (4i) and then in (3) to find p and q.

The maximum allowable value of V consistent with a non J or U shaped beta distribution is $V = \frac{1}{12}$ corresponding to the uniform distribution. For this value of V the constant term of (5) is zero, and consequently A = 0 is a solution of (5). From (4) this implies that B = 0, and hence the required uniform distribution values p = 1, q = 1 are implied.

It is now shown that for $0 < V < \frac{1}{12}$ equation (5) has exactly one positive root. Over this domain the first and last terms of

(5) are always respectively positive and negative. Thus, the number of variations in sign of the coefficients of (5) depend on the inter two terms. The possible number of variations in sign are given as follows.

		$7V - M + M^2$ +	$7V - M + M^2$ −
$16V - 1$	+	1	3
	−	1	1

Now $16V - 1 > 0$ and $7V - M + M^2 < 0$ imply $1/16 < V < \frac{M - M^2}{7}$ or $7/16 < M - M^2$. However, $M - M^2$ has a maximum value of 1/4 at $M = 1/2$, and consequently three variations in sign are not possible for $0 < V < 1/12$. Therefore, by an extension to Descartes' rule of sign there must be exactly one positive root (see for example, Richardson 1947, pp. 240-243).

Approximation procedures for fitting a beta distribution using the mode and variance have been developed for use with PERT project control. These procedures usually assume that $\sigma = (b-a)/6$. With this assumption a linear approximation, $p/(p+q) = (a+4 \text{ Mode}(e\beta)+b)/6$, can be used. This avoids solving the cubic equation (5). The effect of these and other beta distribution assumptions are discussed by MacCrimmon and Ryavec (1964).

A2.5.2 An Information Ratio for Comparing Beta Prior Probability Density Functions

In evaluating alternative prior p.d.fs. it is convenient to have a measure of the impact of one prior p.d.f. relative to another alternative prior p.d.f. Draper and Guttman (1969) have proposed that the ratio of the expected values of the posterior variance of the unknown parameter with respect to a given sample size be used as such a measure.

For a standardized beta p.d.f., $f(t|r',n'-r')$, (A2.2(1) with $a = 0$ and $b = 1$) and for a sample size n with random outcome r Draper and Guttman have shown that the

$$E_r(\mathrm{Var}(t|r,n)) = \frac{n'(1-\mu_1')\mu_1'}{(n'+1)(n+n')}$$

where the E_r is the expectation with respect to the preposterior distribution of r.

Now if the mean, μ_1', of an alternative prior is the same, but the information content is kn' rather than n', the information ratio is*

$$R_n = \frac{E_r(\mathrm{Var}(t|r,n)|r',n'-r')}{E_r(\mathrm{Var}(t|r,n)|r',kn'-r')}$$

$$= \frac{1}{k}\,\frac{(kn'+1)(n+kn')}{(n'+1)(n+n')} \tag{6}$$

$$\tilde{R}_n \cong \frac{n+kn'}{n+n'} \qquad \text{for small k relative to } kn' \tag{7}$$

Thus, if a competitive prior represents k times as many equivalent sample items as n', the expected relative reduction in the posterior variance with a sample of n is smaller than k. For example, given $n' = 60$, $k = 3$ and $n = 100$ it follows from (6) and (7) that $R_n = 1.73$ and $\tilde{R}_n = 1.75$. For $n' = 120$, $\tilde{R}_n = 2.09$.

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*Draper and Guttman define k slightly different. Our formulation leads, to the customary interpretation of equivalent prior sample and simplifies the final result.

A2.5.3 Tail Probabilities for the Standardized Beta Distribution

In evaluating a prior beta p.d.f. it is useful to determine tail positions of the distribution. In particular, one often wishes to fix α and solve for t_α in the equation

$$F_\beta(t_\alpha|p,q) = \int_0^{t_\alpha} \frac{t^{p-1}(1-t)^{q-1}}{B(p,q)}\,dt = 1 - \alpha \tag{8}$$

Such inversion procedures are usually included in a computer statistical package of subroutines. However, in the author's experience they are not always designed for large values of q. Two recent approximations by Boyd (1971) and by Cole (1975) (see also, Bird Engineering 1974, pp. 9-8 to 9-10) provide simple computationally efficient alternatives to a "canned" program.

Both procedures are motivated by reliability analyses and consequently assume that the probability mass of the distribution is concentrated near one. These conditions can be generated in a low error rate environment with the change of variable R = 1 - t to (8). This yields

$$F_\beta(t_\alpha|p,q) = P[\tilde{R}>R_\alpha] = \int_{R_{1-\alpha}=1-t_\alpha}^{1} f_\beta(R|q,p)dR = 1 - \alpha \tag{9}$$

Both procedures are now restated in terms of the parameterization given in (9). Because of the way the approximations have been developed they should not be applied directly to (8).

Cole's approximation is based upon an iterative search for integer i using A2.2(1) such that $\mu_i'(s) \cong E(R^i) \cong 0.5$. It is then assumed that R^i is normally distributed with mean and variance of

$$\mu_1'(R^i) = \mu_i'(s\beta) \qquad \mu_2(R^i) = \mu_{2i}'(s\beta) - \mu_i'(s\beta)^2 \tag{10}$$

This implies that

$$P(R \geq [Z_{1-\alpha}\mu_2(R^i)^{1/2} + \mu_1'(R^i)]^{1/i}) \cong 1 - \alpha$$

where $Z_{1-\alpha}$ is a unit normal lower tail value such that $P(Z \geq Z_{1-\alpha}) = 1-\alpha$. Substituting from (9) and (10) the desired solution to (8) is then

$$t_\alpha = 1 - [Z_{1-\alpha}(\mu_{2i}'(s) - \mu_i'(s)^2)^{1/2} + \mu_i'(s)]^{1/i} \tag{11}$$

where the moments are with respect to $f_\beta(t|q,p)$ rather than $f_\beta(t|p,q)$.

Boyd expressed a cumulative beta distribution as a cumulative binomial distribution (Raiffa and Schlaifer 1961, p. 217) and then used the arcsin normal approximation to the binomial distribution. For $f_\beta(R|q,p)$ this leads to 2 arcsin$\sqrt{R}$ being approximately normally distributed* with mean and variance of

$$\mu_1'(2\arcsin\sqrt{R}) = 2\arcsin\left(\frac{q+1/2}{p+q-1}\right)^{1/2} \quad \mu_2(2\arcsin\sqrt{R}) = (p+q-1)^{-1} \tag{12}$$

Using (12), equation (9) can be written as

$$P(\tilde{R} \geq R_{1-\alpha}) = P\left(\tilde{R} \geq \left[\sin^2\left(\frac{Z_{1-\alpha}}{2(p+q-1)} + \arcsin\left(\frac{q+1/2}{p+q-1}\right)^{1/2}\right)\right]\right) = 1 - \alpha$$

Boyd makes some computationally oriented adjustments to this result. This leads to the approximation

$$t_\alpha = 1 - R_{1-\alpha} = 1 - \left[\sin^2\left(\frac{Z_{1-\alpha}}{2(p+q-1)^{1/2}} + \arcsin\left(\frac{q}{p+q}\right)^{1/2}\right) + \frac{1}{3(p+q-1)}\right] \tag{13}$$

---

*Boyd concluded that 2 arcsin R is normally distributed (p. 12). This expositional error does not affect his subsequent analysis or results.

Boyd reported that for values of $q > 40$ the worst estimate with (13), observed over a wide range of p and q, was less than .003 away from the true value. He also gives several alternative forms of the approximation and discusses their use. Cole does not comment on the accuracy of his approach. Unlike Boyd's procedure it is only valid for unimodal distributions for which $p + q > p + 1 > 2$.

A2.5.4 A Survey of the Literature on the Product of Beta Distributions

The distribution of the product of independent beta random variables has been studied by numerous authors. This work has been motivated in part by the Bayesian formulation of reliability theory, where the product of r.vs. arise in the analysis of series and parallel systems of components. The most complete analysis to date is given by Springer and Thompson (1970). They developed a closed form expression for the product of independent, nonidentically distributed beta r.vs. with integer parameters. This work encompasses the earlier work by Lomnicki (1967) and by Springer and Thompson (1966a,b) for the product of independent identically distributed r.v.

Several difficulties may arise in using the Springer and Thompson (1970) results for a product of beta r.vs. These results are extremely intricate and difficult to compute for the highly skewed p.d.f. that can arise in auditing work. The computer processing time required to compute these results could limit their use in a sensitivity analysis of an input parameter set. Further, it would be very difficult to determine the moments of the product p.d.f. for input into subsequent steps.

Particularly relevant to the focus of this appendix is the work of Jambunathan (1954), and Garner (1969). Jambunathan showed that for certain combinations of parameters the product of beta p.d.fs. is also a beta p.d.f. Garner independently derived some of the results given by Springer and Thompson (1970). Of particular interest is his alternative derivation of the distribution of a product of beta random variables. This approach clearly demonstrates that the product of beta r.vs. can be expressed as an infinite mixture of beta p.d.fs. This is the same general form as a Jacobi orthogonal expansion.

Garner also found that 95% confidence intervals for the product of two moderately skewed beta r.vs. could be approximated quite accurately by a single beta p.d.f. with the same first two moments. Similar results for the product of 15 and 25 highly skewed beta r.vs. were observed by Foard (1971).

Bird Engineering (1974) has also investigated this issue. They state without supporting details that "After comparing the plots of these exact curves with the beta fits to the first two moments for many examples, it was concluded that there was not sufficient differences to ever warrant the use of these highly untractable curves (the Springer and Thompson results)" (pp. 9-23).

A2.5.5 Prior to Posterior Analysis for a Bernoulli Process Using Only Moments

An easily derived result due to Mastran (1976) is now developed. Applying Bayes' law to a binomial sample of r errors in n trials it follows for any prior p.d.f. f(ρ) that

$$E(\rho^m|r,n) = \int_0^1 \rho^m[k\rho^r(1-\rho)^{n-r}f(\rho)]d\rho \tag{14}$$

where the [...] term is the posterior p.d.f. of ρ with normalization constant k.

On expanding $(1-\rho)^{n-r}$ (14) reduces to

$$E(\rho^m|r,n) = K \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} E(\rho^{m+r-i}) \tag{15}$$

where the latter expectation is with respect to the prior p.d.f. The constant k is determined by setting m = 0 in (15).

For example for r = 2, n = 100 and m = 4 equation (15) requires the calculation of $E(\rho^6)$ to $E(\rho^{104})$. In order to develop some insight about the numerical significance of these terms a beta prior of $r' = 3$ and $n' = 60$ was used in this example. This led to numerically significant terms given by $E(\rho^{15})$ to about $E(\rho^{45})$.

In order to avoid problems of numerical precision it is helpful if such higher moments can be readily calculated. The lognormal distribution is such an example. In this case

$$\log E(\rho^{m+r+i}) = (m+r+i)\xi + \frac{1}{2}(m+r+i)^2\sigma^2$$

where ξ,σ are respectively scale and slope parameters (Johnson and Kotz 1970a, p. 115).

A2.5.6 Scaling an Extended Beta Distribution

This section demonstrates that X = wT and Y = wT(w > 0) are both extended beta r.vs. when T also is. Thus the extended beta distribution is closed with respect to any positive or negative scaling. The first case with w > 0 is quite routine. Applying this change of variable to A2.1(1) yields the p.d.f.

$$f(x) = \frac{(\frac{x}{w} - a)^{p-1}(b - \frac{x}{w})^{q-1}}{B(p,q)(b-a)^{p+q+1}} \frac{1}{w} = f_\beta(x|p,q,aw,bw)$$

where $aw \leq x \leq bw$ $\quad p > 0, \quad q > 0$

In the second case where $-w < 0$ it follows that

$$f(y) = \frac{(\frac{-y}{w} - a)^{p-1}(b - \frac{-y}{w})^{q-1}}{B(p,q)(b-a)^{p+q-1}} \frac{1}{w} = \frac{[y-(-bw)]^{q-1}[(-aw)-y]^{p-1}}{B(q,p)[(-aw)-(-bw)]^{p+q-1}}$$

$$= f_\beta(y|q,p,-bw,-aw) \qquad -bw \leq y \leq -aw$$

APPENDIX 3

ORTHOGONAL EXPANSIONS USING JACOBI POLYNOMIALS

A3.1 Preface

In the development of the evidential integration model several mathematical difficulties arise. It is possible to determine the moments of several unknown p.d.fs. of the model but not closed form expressions for their exact p.d.fs. These inconveniences can be resolved using an "orthogonal expansion" for the unknown p.d.fs. This expansion expresses an unknown p.d.f. as an infinite series of polynomials based upon the known moments and a "weighting function." Since these summary moments can arise out of correlated functions of r.vs., an orthogonal expansion permits a natural extension to models of statistically dependent physical processes.

The following type of orthogonal expansion is investigated in this appendix

$$f(x) = \sum_{n=0}^{\infty} C_n w(x) P_n^{\alpha,\beta}(x) \qquad (1)$$

In this expression w(x) is a weighting function proportional to a beta p.d.f., $P_n^{\alpha,\beta}(x)$ is a Jacobi polynomial with n terms and C_n are constants that depend upon the moments of the unknown p.d.f. The parameters $\alpha > -1$ and $\beta > -1$ can be arbitrarily chosen. This freedom is particularly important with truncated forms of (1).

In the contents of this dissertation the Jacobi "system" of orthogonal polynomials offers several attractive features. The Jacobi expansion is analytically compatible with the "beta-normal" p.d.f. technique of Felix and Grimlund (1977) for combining error rate and error size data. Further, there is reason to believe that a parsimonious approximation to the unknown p.d.fs. can be achieved.

While the possibility of a Jacobi expansion for an unknown p.d.f. is generally recognized, very little has been written about the expansion. There is an application by Durban and Watson (1951) and several brief references (Kendall and Stuart 1958, p. 163; Wolf 1976). Pinney (1947) has developed some of the background mathematics. There does not appear to be a unified development of the subject such as provided by this appendix.

After introducing the auditing problems that motivate this research, the discussion turns briefly to the current status of research in several areas of mathematical statistics which motivate the approach of this appendix. Next, several potential problems with orthogonal expansions are discussed. These problems have motivated this dissertation exploration of the Jacobi expansion. After providing in section A3.3 an overview of the theory of Jacobi orthogonal polynomials, section A3.4 develops the specific details for the Jacobi expansion of p.d.fs. Section A3.5 analyzes the customary "Edgeworth expansion" for the "Gram-Charlier" orthogonal series, and shows how it might be applied to the Jacobi expansion.

A3.2 Motivational Issues

In order to motivate the extensive technical development of this appendix two possible auditing uses for an orthogonal expansion are now briefly discussed.

A3.2.1 The Auditing Problem

The possible auditing applications of orthogonal expansions pertain to the analyses of i.c.ss., and the consolidation of the total error uncertainty for several stratum of accounts.

When posting entries for a given account are generated out of several i.c.ss. one expects that the auditor's expression of his total uncertainty in the account error rate will be a sum of products of r.vs. Each i.c.s. can be considered as a separate application of reliability theory leading to a composite error rate r.v. These i.c.s. composite r.vs. are then combined into a weighted sum.

Several authors have suggested that high skewed and leptokurtic beta distributions could be used by the auditor to represent his uncertainty for error rates (see Felix, 1976; Felix and Grimlund 1977; Francisco 1972). Since sample error rates observed in auditing typically are extremely small, the posterior p.d.fs. of these error rates should also be very leptokurtic and skewed. Further, one does not expect that there will be a sufficient number of i.c.s. to generate a usable central limit theorem effect when forming a weighted sum of i.c.s.

Similar observations can be made for the sum of r.vs. used to consolidate the total dollar error of several strata of accounts. It is shown in section A4.3 that the beta-normal p.d.f. suggested by Felix and Grimlund (1977) for representing the total dollar error in an account stratum can be approximated by a skewed extended beta p.d.f. with range [a,b]. In this second application one expects that the unknown summary p.d.f. will be a sum of skewed beta p.d.fs.

For both of these applications, the sum of products of beta r.vs. and the sum of beta r.vs., there appears to be no published statistical research that directly considers beta distributions. In these cases the usual integral transformation theory (i.e., Characteristic Function, Mellin transformation, etc.), fails to yield tractable results. The only aspect of these steps for which a statistical theory has been developed is with the product of beta distributions (see paragraph A2.5.4).

A3.2.2 The Choice of the Jacobi Orthogonal System

When using an orthogonal expansion for specifying an unknown p.d.f. it is often advisable to avoid using higher order moments. Problems can arise because of the extreme numerical sensitivity of the fifth and higher sample moments to single large observations. There also is a problem of developing an orthogonal expansion with multimodal behavior and possible negative ranges for the approximate p.d.f. It is generally believed that these difficulties become more prevalent when a large number of terms are used (see Johnson and Kotz 1970a, pp. 19,35; Kendall and Stuart 1958, pp. 159-161).

When the expansion will be used in subsequent analytical steps of an analysis it can be highly desirable to have a compact representation with only a few terms. These issues must be considered when probability distributions and logical relationships are available for generating higher moments. In such situations, it is possible in theory to compute additional terms of the expansion. However, these empirical convenienciences do not necessarily justify the inclusion of the additional terms.

As discussed by Pearson (1963) the higher moments are predominately determined by the extended right tail of a leptokurtic and positively skewed distribution. Thus, besides the availability of these higher moments one must consider the reliability of the tail probabilities of the source distributions.

These comments are particularly appropriate to the development of auditing models of i.c.ss. When Bayesian derived distributions with significant prior judgments are used to represent the error rates within process steps, the reliability of the right tail probability must be considered. The exact nature of this p.d.f. tail may be just a byproduct of the judgmental specification procedures. These procedures may not emphasize the tail areas over which the decision maker is liable to have very little experience. In such cases the orthogonal expansion terms that are based upon higher moments are likely to have very little informational content.

A major consideration in constructing an orthogonal expansion for a p.d.f. is the previously mentioned possibility of generating multimodal expansions with regions of negative probability mass. These problems have been empirically studied by Barton and Dennis (1952), and Berndt (1957) for the "Edgeworth" and "Gram-Charlier" series. These series are two forms of orthogonal expansions based upon a normal p.d.f. for weighting the associated "Hermite" polynomials. This work showed that in these Hermite expansions such problems are apt to arise when approximating skewed, leptokurtic distributions such as can be encountered in a Bayesian reliability model.

There are further questions of convergence of Hermite expansions when the unknown distributions are far from normal (see Cramer 1946, pp. 223-224; Kendall and Stuart 1958, pp. 161-163; Pearson 1963, p. 95;

Wallace 1958, pp. 635-636). The expansion is often used when the somewhat restrictive sufficient conditions for convergence are not satisfied, or when the rate of convergence inhibits the development of a parsimonious expression for the approximation. When such conditions are relevant one is interested in the performance of the initial terms of the expansion, rather than the asymptotic properties of the series.

The above discussion suggests that it is best to match as near as possible the p.d.f. of the weighting component of an orthogonal expansion to the anticipated form of the unknown p.d.f. The highly leptokurtic and skewed distributions of an audit model of an i.c.s. suggest that a beta type Jacobi expansion may be more appropriate. This is a particularly appealing choice since the beta p.d.fs. can also "model" the more modest leptokurtic and skewness structure that will arise when a number of these extreme r.vs. are multiplied or added together. Finally, as is discussed in paragraph A3.4.2 the Jacobi polynomial expansion will usually lead to a uniform convergent series.

A3.3 The Jacobi Orthogonal Polynomial System

Orthogonal polynomials are useful in a wide variety of applications such as numerical integration, physical systems with partial differential equations and probability analysis. There are several systems of orthogonal polynomials, each arising as a solution to a particular family of second order differential equations. Beckmann (1973) has developed a very readable and timely survey of the general properties and interrelationships between different types of orthogonal polynomials.

It should be emphasized that orthogonal polynomials are nothing more than ordinary polynomials with the coefficients adaptly chosen so as to satisfy certain useful relationships. For each orthogonal system, there is a Taylor series expansion of a "generating function" which will yield all the polynomials of the system as the coefficients of the terms of the Taylor series. In addition the degree n polynomial of such a system is the n^{th} derivative of the system's "characterizing function." Further, all the polynomials of a system satisfy an "orthogonalizing integral relationship," and are functionally related to a hypergeometric function (see Bell 1968, p. 204).

A3.3.1 A Survey of Properties

The hypergeometric function is often used to define a system of orthogonal polynomials. In particular the Jacobi orthogonal polynomial of degree n with parameters α and β can be defined as

$$P_n^{\alpha,\beta}(x) = \binom{n+\alpha}{n} {}_2F_1(-n, n+\alpha+\beta+1, \alpha+1; \tfrac{1-x}{2}) = \sum_{r=0}^{n} A_r^{(n)} \left(\frac{x-1}{2}\right)^r$$

$$\text{where } A_r^{(n)} = \frac{1}{n!}\binom{n}{r}\frac{\Gamma(n+\alpha+1)\Gamma(n+r+\alpha+\beta+1)}{\Gamma(\alpha+r+1)\Gamma(n+\alpha+\beta+1)} \qquad (1)$$

The notation ${}_2F_1$ is used to indicate the Gaussion hypergeometric function, an important member of the generalized family of hypergeometric functions denoted by pFq (see Bell 1968, pp. 199,203-217; or Szego 1939, pp. 62-63).

Particularly relevant to this dissertation is the following orthogonal property of Jacobi polynomials (Bell 1968, p. 199).

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}\, P_n^{\alpha,\beta}(x)\, P_m^{\alpha,\beta}(x)\, dx = h_n^{\alpha,\beta} 2^{\alpha+\beta+1} \delta_{n,m} \qquad (2)$$

where

$$h_n^{\alpha,\beta} = \frac{1}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)} \qquad \alpha > -1 \quad \beta > -1 \tag{3}$$

$$\delta_{n,m} = 1 \quad \text{if } n = m$$

$$= 0 \quad \text{if } n \neq m$$

Thus, the system of Jacobi polynomials $\{P_n^{\alpha,\beta}(x)\}$ are orthogonal over the interval [-1, 1] with respect to the weighting function $w(x) = (1-x)^\alpha(1+x)^\beta$ and norm $h_n^{\alpha,\beta}\ 2^{\alpha+\beta+1}$. These three attributes, an interval, weighting function and norm, completely characterize a system of orthogonal polynomials. When a change of variable is applied to (2) to shift or expand the interval, the resulting polynomials are at times called shifted Jacobi polynomials.

There are only three combinations of intervals and weighting functions for which there exists an equation similar to (2). These three basic systems correspond to finite intervals, bounded from below or above intervals, and an unbounded interval. These lead respectively to the Jacobi, Laquerre and Hermite systems of polynomials each with an associated weighting function. There are numerous special cases of the Jacobi polynomials known as various types of Legendre and Chebyshev polynomials (see Beckmann 1973, pp. 65-66,76-78).

A3.3.2 The Jacobi Expansion

The orthogonal property (2) allows one to determine the coefficients of an orthogonal expansion of a known continuous function. Assuming uniform convergence for $\alpha > -1$, $\beta > -1$ of $g(x) = \sum_{i=1}^{\infty} C_i P_i^{\alpha,\beta}(x)$ (Rau 1949-1950), it follows upon multiplying both sides by $w(x)P_n^{\alpha,\beta}(x)$ and integrating from -1 to 1 that

$$C_n = \frac{1}{h_n^{\alpha,\beta} 2^{\alpha+\beta+1}} \int_{-1}^{1} w(x)\ P_n^{\alpha,\beta}(x) g(x) dx$$

An unknown p.d.f. f(x) with known noncentral moments, $\mu_r(x)$, can be expanded using a slightly modified procedure. For example, for $x\varepsilon[0,1]$ it can be shown for $\alpha > -1$, $\beta > -1$ that

$$f(x) = 2^{-(\alpha+\beta)} w(1-2x) \sum_{i=0}^{\infty} C_i\ P_i^{\alpha,\beta}(1-2x) \tag{4}$$

where

$$C_n = \frac{1}{h_n^{\alpha,\beta}} \int_0^1 P_n^{\alpha,\beta}(1-2x) f(x) dx$$

$$2^{-(\alpha+\beta)} w(1-2x) = x^{\alpha}(1-x)^{\beta}$$

Since $P_n^{\alpha,\beta}(1-2x)$ is a polynomial of degree n, the r.h.s. integral of (4) can be integrated termwise using (1). This yields $C_n = D_n/h_n^{\alpha,\beta}$ where

$$D_n = \sum_{r=0}^{n} (-1)^r\ A_r^{(n)} \mu_r(x)$$

For all three orthogonal systems there is an alternative formulation to the type of expansion given in (4). This is based upon the following generalized Rodriquez formula (Beckmann 1973, p. 47).

$$w(y)Q_i(y) = A_i\left(\frac{d}{dy}\right)^i [w(y)B^i(y)] \tag{5}$$

For Jacobi polynomials one has

$$\begin{aligned} Q_i(y) &= P_i^{\alpha,\beta}(y) & \qquad w(y) &= (1-y)^{\alpha}(1+y)^{\beta} \\ A_i &= (-1)^i/2^i i! & \qquad B(y) &= (1-y^2) \end{aligned} \tag{6}$$

Letting y - 1-2x and noting for this changing variable that

$$(\frac{d}{dy})^i f(y) = (-2)^i \frac{d}{dx} f(1-2x)$$

it follows from (5) and (6) that (4) can be written as

$$f(x) = \sum_{i=0}^{\infty} \frac{1}{i!} C_i (\frac{d}{dx})^i [x^{\alpha+i}(1-x)^{\beta+i}] \tag{7}$$

Since the [...] term is the kernel of a beta p.d.f., the expansion for f(x) is a linear weighting of derivatives of this density. In paragraph A3.4.4 it is shown that a truncated version of the expansion is a weighted linear function of beta p.d.fs.

A3.4 The Jacobi Expansion for an Unknown p.d.f.

Orthogonal expansions of unknown p.d.fs. are discussed by Khamis (1958) for Laquerre polynomials with gamma probability density weighting functions, and by Kendall and Stuart (1958, pp. 155-163) for Hermite polynomials with normal probability weighting functions. For Jacobi polynomials there has been some work done by Pinney (1947).

Pinney developed a moment oriented curve fitting technique which is in fact a truncated form of a Jacobi expansion based upon a beta probability density weighting function. Pinney derived a formula for fitting a curve defined on [0,1] to a set of N moments. In the process of deriving his results he drew upon the properties of Jacobi polynomials. However, he did not emphasize the orthogonal basis of the expansion. There also are some fine points in his results that can be overlooked when routinely attempting to extend the procedure to an arbitrary finite interval. Consequently Pinney's results are extended in this section to the interval (a,b) using the infinite orthogonal expansion framework of this chapter.

A3.4.1 The Development of the Expansion

Let f(x) be an unknown p.d.f. defined on the interval (a,b) with known noncentral moments $\mu_i(x)$ for i = 0, 1, 2,... Assume that for appropriately chosen constants C_k, there exist a uniformly convergent sequence such that

$$f(x) = 2^{-(\alpha+\beta)} w(1-2(\tfrac{x-a}{b-a})) \sum_{n=0}^{\infty} C_n \; P_n^{\alpha,\beta}(1-2(\tfrac{x-a}{b-a})) \tag{1}$$

where $\{P_n^{\alpha,\beta}(y)\}$ and $w(y) = (1-y)^{\alpha}(1+y)^{\beta}$ are the usual Jacobi polynomial system and weighting functions defined on [-1, 1].

If (1) holds then the constants C_k can be determined using the orthogonal properties of $\{P_n^{\alpha,\beta}(y)\}$. Upon multiplying by $P_k^{\alpha,\beta}(1-2(\frac{x-a}{b-a}))$ and integrating from a to b (1) becomes

$$\int_a^b P_k^{\alpha,\beta}(1-2(\tfrac{x-a}{b-a}))f(x)dx = 2^{-(\alpha+\beta)} \sum_{n=0}^{\infty} C_n \int_a^b w(1-2(\tfrac{x-a}{b-a}))\, P_k^{\alpha,\beta}(1-2(\tfrac{x-a}{b-a})) \cdot P_n^{\alpha,\beta}(1-2(\tfrac{x-a}{b-a}))dx \tag{2}$$

With the change of variable $y = 1-2(\frac{x-a}{b-a})$ to the r.h.s. integral, it follows from A3.3(2) that the r.h.s. of (2) is equal to

$$\frac{b-a}{2^{\alpha+\beta+1}} \sum_{n=0}^{\infty} C_n \int_0^1 w(y)\; P_k^{\alpha,\beta}(y)\; P_n^{\alpha,\beta}(y) = (b-a)h_k^{\alpha,\beta}C_k$$

Now the l.h.s. of (2) can be written as

$$P_k^{\alpha,\beta}(1-\mu(\tfrac{x-a}{b-a}))$$

where this expression symbolizes the linear function of moments formed by expanding $P_k^{\alpha,\beta}(1-2(\frac{x-a}{b-a}))$ into a polynomial in $(\frac{x-a}{b-a})$ and replacing $(\frac{x-a}{b-a})^r$ for r = 1, 2,..., k by

$$\mu_r(\frac{x-a}{b-a}) = \int_a^b (\frac{x-a}{b-a})^r f(x)dx = (\frac{1}{b-a})^r \sum_{i=0}^{r} (-1)^i \binom{r}{i} a^i \mu_{r-i}(x) \tag{3}$$

Combining these results, it follows from (2) that

$$C_k = \frac{P_k^{\alpha,\beta}(1-\mu(\frac{x-a}{b-a}))}{(b-a)h_k^{\alpha,\beta}} \tag{4}$$

From (1), (4) and A3.3(3) the Jacobi expansion is given by

$$f(x) = \bar{w}(x) \sum_{n=0}^{\infty} \frac{D_n}{h_n^{\alpha,\beta}} P_n^{\alpha,\beta}(1-2(\frac{x-a}{b-a})) \tag{5}$$

where

$$\bar{w}(x) = (x-a)^{\alpha}(b-a)^{\beta}/(b-a)^{\alpha+\beta+1}$$

$$D_n = P_n^{\alpha,\beta}(1-2\mu(\frac{x-a}{b-a}))$$

$$h_n^{\alpha,\beta} = \frac{1}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}$$

Note that $\bar{w}(x)/B(\alpha+1, \beta+1)$ is a beta p.d.f. defined on [a,b].

From A3.3(1) it follows that

$$P_n^{\alpha,\beta}(1-2(\frac{x-a}{b-a})) = \sum_{r=0}^{n} (-1)^r A_r^{(n)} (\frac{x-a}{b-a})^r \tag{6}$$

where

$$A_r^{(n)} = \frac{1}{n!} \binom{n}{r} \frac{\Gamma(n+\alpha+1)\Gamma(n+r+\alpha+\beta+1)}{\Gamma(\alpha+r+1)\Gamma(n+\alpha+\beta+1)}$$

Combining (5) and (6) leads to

$$f(x) = \bar{w}(x) \sum_{n=0}^{\infty} a_n \tag{7}$$

where

$$a_n = \sum_{r=0}^{n} (-1)^r \, a_{n,r} \left(\frac{x-a}{b-a}\right)^r \quad \text{with} \quad a_{n,r} = \frac{A_r^{(n)} D_n}{h_n^{\alpha,\beta}}$$

From (6) it follows that

$$D_n = P_n^{\alpha,\beta}\left(1-2\mu\left(\frac{x-a}{b-a}\right)\right) = \sum_{r=0}^{n} (-1)^r \, A_r^{(n)} \mu_r\left(\frac{x-a}{b-a}\right) \tag{8}$$

Equations (7) and (8) with $a = 0$ and $b = 1$ correspond to the results given by Pinney for a finite sum.

A3.4.2 Uniform Convergence

The analysis of the last paragraph has assumed that (1) is uniformly convergent to the unknown p.d.f. A theorem due to Rau (1949-1950) is now used to show that a numerically insignificant modified version of (1) is uniformly convergent for all continuous $f(x)$ defined on $[a,b]$. Rau showed that the series $\sum_{n=0}^{\infty} (C_n/2^{\alpha+\beta+1}) P_n^{\alpha,\beta}(y)$ with

$$C_n = \frac{1}{h_n^{\alpha,\beta}} \int_{-1}^{1} g(y) w(y) P_n^{\alpha,\beta}(y) \, dy \qquad w(y) = (1-y)^{\alpha}(1+y)^{\beta} \tag{9}$$

is uniformly convergent on $(-1, 1)$ to $g(y)$, provided that $g(y)$ is continuous on $[-1, 1]$ with at least piecewise continuous derivatives.

In order to apply Rau's theorem to (1) note that with the change of variable $y = 1-2\left(\frac{x-a}{b-a}\right)$, equation (1) can be expressed for $y \varepsilon (-1, 1)$ as

$$g(y) = \frac{f\left((1-y)(b-a)/2+a\right)}{w(y)(b-a)} = \sum_{n=0}^{\infty} (C_n/2^{\alpha+\beta+1}) P_n^{\alpha,\beta}(y) \tag{10}$$

where

$$C_n = \frac{1}{h_n^{\alpha,\beta}} \int_{-1}^{1} \frac{f((1-y)(b-a)/2+a)}{(b-a)} P_n^{\alpha,\beta}(y)dy \tag{11}$$

The series (10) can be modified slightly to conform to the requirements of Rau's uniform convergence theorem. The weighting function $w(y)$ is replaced by $w_\varepsilon(y) = (1-y+\varepsilon)\ (1+y+\varepsilon)^\beta$ for $\varepsilon > 0$. The coefficient C_n is replaced by

$$C_n^{(\varepsilon)} = \frac{1}{h_n^{\alpha,\beta}} \int_{-1}^{1} \left(\frac{f((1-y)(b-a)/2+a)}{w_\varepsilon(y)(b-a)}\right) w(y) P_n^{\alpha,\beta}(y)dy \tag{12}$$

Now the new series for $g_\varepsilon(y)$ defined by the [...] term of (12) satisfies the conditions of Rau's theorem and is uniformly convergent in (-1, 1). Now since (12) is true for all $\varepsilon > 0$, $w(y)/w_\varepsilon(y)$ can be made arbitrarily close to 1, and hence $C_n^{(\varepsilon)}$ of (12) can be made arbitrarily close to C_n of (11). Similarly $w_\varepsilon(y)$ can be made arbitrarily close to $w(y)$. Thus, to any desired degree of numerical precision $g(y)$ of (10) is equivalent to the uniformly convergent series $g_\varepsilon(y)$.

A3.4.3 Properties of the Truncated Series

Pinney showed that a truncated form of (7) with $a = 0$ and $b = 1$ has the same moments as used by (8) in constructing (7). The subsequent discussion for the arbitrary interval [a,b] closely follows Pinney's derivation, and leads to the same conclusion. It is assumed that the moments $\mu_n(\frac{x-a}{b-a})$, $n = 0, 1, \ldots, N$ are known and that $\mu_0 = 1$.

The derivation is based upon the observation that for $0 \leq m \leq N$ there exists constants $A_{m,k}$ $(k = 0, 1, \ldots, N)$ such that

$$\left(\frac{x-a}{b-a}\right)^m = \sum_{k=0}^{m} A_{m,k}\, P_k^{\alpha,\beta}\left(1-2\left(\frac{x-a}{b-a}\right)\right) \tag{13}$$

For example for m = 2, in the notation of (6), equation (13) implies that

$$\left(\frac{x-a}{b-a}\right)^2 = A_{2,0}[A_0^{(0)}] + A_{2,1}[A_0^{(1)}-A_1^{(1)}\left(\frac{x-a}{b-a}\right)]$$

$$+ A_{2,2}[A_0^{(2)}-A_1^{(2)}\left(\frac{x-a}{b-a}\right) + A_2^{(2)}\left(\frac{x-a}{b-a}\right)^2]$$

where the [...] terms represent the orthogonal polynomials of $P_k^{\alpha,\beta}(1-2(\frac{x-a}{b-a}))$. For this case the appropriate constants could be found by solving the following linear system.

$$\begin{aligned} A_0^{(0)} A_{2,0} + A_0^{(1)} A_{2,1} + A_0^{(2)} A_{2,2} &= 0 \\ A_1^{(1)} A_{2,1} + A_1^{(2)} A_{2,2} &= 0 \\ A_2^{(2)} A_{2,2} &= 1 \end{aligned}$$

By the nature of the way the constants are determined, it follows that

$$\mu_m\left(\frac{x-a}{b-a}\right) = \sum_{k=0}^{m} A_{m,k}\, P_k^{\alpha,\beta}\left(1-2\mu\left(\frac{x-a}{b-a}\right)\right) \tag{14}$$

Defining $f_{TN}(x)$ as a truncated version of (5) with N terms it follows from (13) that

$$\int_a^b \left(\frac{x-a}{b-a}\right)^m f_{TN}(x)\,dx = \tag{15}$$

$$\int_a^b \left[\sum_{k=0}^{m} A_{m,k}\, P_k^{\alpha,\beta}\left(1-2\left(\frac{x-a}{b-a}\right)\right)\right]\left[\bar{w}(x) \sum_{n=0}^{N} \frac{D_n}{h_n^{\alpha,\beta}}\, P_n^{\alpha,\beta}\left(1-2\left(\frac{x-a}{b-a}\right)\right)\right]dx$$

The r.h.s. of (15) can be simplified by making the change of variable $y = 1-2(\frac{x-a}{b-a})$ and then applying the orthogonal relationships given by A3.3(2). This leads to

$$\int_a^b (\frac{x-a}{b-a})^m f_{TN}(x)dx = \sum_{k=0}^{m} A_{m,k} D_k = \sum_{k=0}^{m} A_{m,k} P_k^{\alpha,\beta}(1-2\mu(\frac{x-a}{b-a})) = \mu_m(\frac{x-a}{b-a}) \quad (16)$$

The latter two equalities follow from (5iii) and (14). Since $0 \leq m \leq n$, (16) implies that the first n moments about a of $f_{TN}(x)$ are equal to the corresponding moments about f(x).

It is now shown that this is also true for the noncentral moments about 0. Equation (3) is first expressed in matrix form for r = 0, 1,..., m. On cancelling out the common (b-a) factors this yields

$$\begin{bmatrix} \mu_1(x-a) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \mu_m(x-a) \end{bmatrix} = \begin{bmatrix} 1-a+0\ldots \\ 1-2a+a^2+0\ldots \\ 1-3a+3a^2-a^3+0\ldots \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \begin{bmatrix} \mu_1(x) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \mu_m(x) \end{bmatrix} \quad (17)$$

As a consequence of (16) the l.h.s. of (17) represents the identical moments of $f_{TN}(x)$ and f(x). Now as a result of the shifted diagonal structure of the m x m matrix of (17), this matrix is of full rank with a unique inverse when $a \neq 0$. Thus (17) can be inverted and used to demonstrate that the r.h.s. column vector is the same for both $f_{TN}(x)$ and f(x). In particular this implies that $\mu_m(x)$ of f(x) is equal to $\mu_m(x)$ of $f_{TN}(x)$, for $0 \leq m \leq N$. Thus the N-moment orthogonal approximation perserves the first N-moments of the target distribution.

A3.4.4 The Selection of α and β and a Linear Function of Beta p.d.fs.

In order to consider how α and β might be selected, it is convenient to express (7) in an alternative form. Observe first that (7) contains a series of polynomials in $y = (\frac{x-a}{b-a})$ of the form

$$a_{0,0} - (a_{1,0}+a_{1,1}y) + (a_{2,0}+a_{2,1}y+a_{2,2}y^2) - \dots \tag{18}$$

Now, when (7) is truncated after n = N, terms of equal power in (18) can be collected together. This yields

$$f_{TN}(x) = \bar{w}(x) \sum_{r=0}^{N} \Big(\sum_{n=r}^{N} (-1)^n a_{n,r} \Big) \big(\frac{x-a}{b-a}\big)^r \tag{19}$$

Recalling from (5) the definition of $\bar{w}(x)$, it follows from (19) that $f_{TN}(x)$ is a linear function of N + 1 beta r.vs. with coefficients (α+1+r, β+1) for r = 0, 1,..., N. Further, if α and β are selected so that

$$a_{N,N} = 0 \qquad \text{and} \qquad a_{N-1,N-1} - a_{N,N-1} = 0 \tag{20}$$

then (19) will have at most N - 1 nonzero terms, and $f_{TN}(x)$ will still have the same moments, $\mu_i(x)$, for i = 0, 1,..., N as f(x). This choice of (α, β) is particularly attractive when a parsimonious approximation of f(x) is desired.

For N = 2, if (20) is satisfied then (19) reduces to

$$f_{T2}(x) = \bar{w}(x)[a_{0,0}-a_{1,0}+a_{2,0}]$$

Now since $f_{T2}(x)$ is a p.d.f., $[a_{0,0}-a_{1,0}+a_{2,0}] = B(\alpha+1, \beta+1)$ and α and β can be determined using the standard formulas of section A2.3.3 for finding the parameters of a beta distribution given its first two moments and [a,b].

Using the definition $a_{n,r} = A_r^{(n)}D_n/h_n^{\alpha,\beta}$ from (7), equation (20) becomes

$$\frac{A_N^{(N)}D_N}{h_N^{\alpha,\beta}} = 0 \quad \text{and} \quad \frac{A_{N-1}^{(N-1)}D_{N-1}}{h_{N-1}^{\alpha,\beta}} - \frac{A_{N-1}^{(N)}D_N}{h_N^{\alpha,\beta}} = 0 \tag{23}$$

Observing that $A_k^{(k)}/h_k^{\alpha,\beta} > 0$ for $k > 0$, (23) reduces to

$$D_N = 0 \quad \text{and} \quad D_{N-1} = 0 \tag{24}$$

Equations (24) can be expressed in terms of α and β as follows.

Defining $S_n = n!D_n$ and letting $\mu_r = \mu_r(\frac{x-a}{b-a})$, it follows from (8) and (6) that

$$S_n = \sum_{r=0}^{n} (-1)^r n!\, A_r^{(n)}\, \mu_r = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\beta+1)} \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{\Gamma(n+r+\alpha+\beta+1)}{\Gamma(\alpha+r+1)}\, \mu_r \tag{25}$$

It follows from (25) that

$$S_0 = 1$$

$$S_1 = (\alpha+1) - (\alpha+\beta+2)\mu_1$$

$$S_2 = (\alpha+2)(\alpha+1) - 2(\alpha+\beta+3)(\alpha+2)\mu_1 + (\alpha+\beta+4)(\alpha+\beta+3)\mu_2$$

and in general

$$S_n = \sum_{i=0}^{n} S_n(1,i)\, S_n(2,i)\, S_n(3,i) \tag{26}$$

where

$$S_n(1) = \begin{bmatrix} (\alpha+n)\ldots(\alpha+1) \\ (\alpha+n)\ldots(\alpha+2) \\ (\alpha+n)\ldots(\alpha+3) \\ \cdot \\ \cdot \\ \cdot \\ (\alpha+n) \\ 1 \end{bmatrix} \qquad S_n(2) = \begin{bmatrix} 1 \\ (\alpha+\beta+n+1) \\ (\alpha+\beta+n+1)(\alpha+\beta+n+2) \\ \cdot \\ \cdot \\ (\alpha+\beta+n+1)\ldots(\alpha+\beta+2n-1) \\ (\alpha+\beta+n+1)\ldots(\alpha+\beta+2n) \end{bmatrix}$$

$$S_n(3) = [\tbinom{n}{o}\mu_0,\ (-1)^1\tbinom{n}{1}\ \mu_1,\ldots,(-1)^n\tbinom{n}{n}\mu_n]$$

Equation (26) provides an efficient way of computing $D_n = S_n/n!$ when calculating $f_N(x)$ or when using an iterative procedure to solve (24) for α and β. In case (24) does not admit a real valued solution with $\alpha > -1$, $\beta > -1$, values for (α,β) can be determined using the procedure previously considered for $N = 2$.

A3.5 An Edgeworth Expansion for a Jacobi Series

In using a truncated orthogonal expansion there is an alternative developed by Edgeworth to simply truncating the series A3.4(1) after the first N terms. While the procedure has been discussed in the literature with normal p.d.f. weightings as an alternative to the Gram-Charlier series, the Edgeworth expansion procedure can be generalized and applied to other orthogonal systems. As will be seen, the procedure may be useful when approximating the p.d.f. of a large sum of somewhat similar independent r.vs.

It is well known that a Gram-Charlier series will not always converge monotonically; that is the order of importance of terms does not correspond to the order in which they are generated. This empirical pattern can arise when the coefficient of the (N+1) term is significantly influenced by the N^{th} and lower moments. Thus, a series truncated with the N^{th} term because the μ_{N+1} moment is unreliable, will not give good results if the coefficient of (N+1) term is dominated by lower moments which are reliable.

A3.5.1 The General Edgeworth Expansion Model

The Edgeworth expansion is based upon a model of how these circumstances could arise. While the assumptions of the model are generally not satisfied, the model can at times provide greater predictability than other relevant alternatives (see Johnson and Kotz 1970a, p. 19). The model is based upon the assumption that the unknown r.v. X is the sum of n independent identically distributed r.vs. Y. Under these circumstances it can be shown that the cumulants, $\kappa_r(z)$ and $\kappa_r(w)$, of the standardized r.vs. $Z = (X - \mu_1(x))/\sigma(x)$ and $W = (Y - \mu_1(y))/\sigma(y)$ are related by $\kappa_r(z) = \kappa_r(w)/n^{(r/2)-1}$ (Cramer 1946, pp. 224-225).

Since $\kappa_r(w)$ is independent of n, each cumulant of z will have a different order of importance. Note that the rankings given by this derivation is only one of many possible rankings that can be created by altering the initial assumptions. Now in general, since each coefficient of an expansion is composed of several cumulants, the coefficient will have several components with different orders of importance. In developing an Edgeworth expansion of order n only those components of order n or less are utilized.

In applying these concepts to the Gram-Charlier series, the mechanics of the analysis tends to obscure the underlying procedure and several theoretical issues that must be considered. While the usual discussion does not emphasize it, the following general steps are taken (see for example Cramer 1946, pp. 221-230).

1. Each coefficient is expressed in terms of cumulants, κ_r, rather than moments, κ_r
2. Each cumulant is replaced by an expression that exhibits its order
3. The expression is truncated at a given order
4. The truncated expressions are converted back to moments which are then used to calculate the Edgeworth coefficients.

Combined formulas for applying steps 1 and 2 can be developed from the central moment to cumulant equations given by Kendall and Stuart (1958, p. 70). Assume as previously indicated that the r.v. X is the sum of n independent r.vs. Y. Now the moments of Z (the standardized form of X) can be expressed in terms of the cumulants W (the standardized form of Y) by substituting Cramer's result, $\kappa_r(z) = \kappa_r(w)/n^{(r/2)-1}$, into the Kendall and Stuart equations. Letting $\kappa_r = \kappa_r(w)$ the resulting formulas for applying steps 1 and 2 are

$$\mu_1(z) = \kappa_1(z) = 0$$

$$\mu_2(z) = \kappa_2(z) = 1$$

$$\mu_3(z) = \frac{\kappa_3}{n^{1/2}}$$

$$\mu_4(z) = \frac{\kappa_4}{n} + 3 \tag{1}$$

$$\mu_5(z) = \frac{\kappa_5}{n^{3/2}} + 10\frac{\kappa_3}{n^{1/2}}$$

$$\mu_6(z) = \frac{\kappa_6}{n^2} + 15\frac{\kappa_4}{n} + 10\frac{\kappa_3^2}{n} + 15$$

$$\mu_7(z) = \frac{\kappa_7}{n^{5/2}} + 21\,\frac{\kappa_5}{n^{3/2}} + 35\,\frac{\kappa_4\kappa_3}{n^{3/2}} + 105\,\frac{\kappa_3}{n^{1/2}}$$

$$\mu_8(z) = \frac{\kappa_8}{n^3} + 28\frac{\kappa_6}{n^2} + 56\frac{\kappa_5\kappa_3}{n^2} + 35\frac{\kappa_4^2}{n^2} + 210\frac{\kappa_4}{n} + 280\frac{\kappa_3^2}{n} + 105$$

In accordance to step 3 an order n expansion is given by tuncating all terms greater than order n in (1). The Edgeworth coefficients of step 4 then are determined by reconverting the remaining cumulants into standardized moments. This leads to approximation equations which can be used in place of higher moments in calculating the coefficients of series expansion. Letting $\mu_i = \mu_i(z)$ it follows that

$$\hat{\mu}_2(z) = 1 \tag{2}$$

$$\hat{\mu}_3(z) = \mu_3$$

$$\hat{\mu}_4(z) = \mu_4$$

$$\hat{\mu}_5(z) = 10\mu_3$$

$$\hat{\mu}_6(z) = 15(\mu_4-3) + 10\mu_3^2 + 15$$

$$\hat{\mu}_7(z) = 105\mu_3$$

$$\hat{\mu}_8(z) = 210(\mu_4-3) + 280\mu_3^2 + 105$$

In addition for $n \geq 9$, it can be shown that

$$\hat{\mu}_n(z) = \frac{n!}{6\cdot(\frac{n-3}{2})!\;2^{(n-3)/2}}\,\mu_3 \quad \text{(n odd)} \tag{3}$$

$$\hat{\mu}_n(z) = \frac{n!}{(\frac{n}{2})!\;2^{n/2}} + \frac{n!}{72\cdot(\frac{n-6}{2})!\;2^{(n-6)/2}}\,\mu_3^2 + \frac{n!}{24\cdot(\frac{n-4}{2})!\;2^{(n-4)/2}}\,(\mu_4-3) \quad \text{(n even)} \tag{4}$$

Equations (3) and (4) are derived by applying an Edgeworth expansion to the equality $\mu_r = f(\kappa_1, \kappa_2, \ldots, \kappa_r)$ given by Kendall and Stuart (1958, p. 68). In our notation for standardized r.vs. with $\kappa_1 = 0$ and $\kappa_2 = 1$, the terms of order n or less of this equality are $\kappa_2^{n/2}$, $\kappa_4 \kappa_2^{(n-4)/2}$ and $\kappa_3^2 \kappa_2^{(n-6)/2}$ for n even, and $\kappa_3 \kappa_2^{(n-3)/2}$ for n odd. Equations (3) and (4) are determined by writing out the coefficients of these terms and setting the standardized cumulants $\kappa_2 = 1$, $\kappa_3 = \mu_3$ and $\kappa_4 = \mu_4 - 3$.

When the order n moments of (2), (3) and (4) are substituted into expressions for the coefficients of the Gram-Charlier series (see Kendall and Stuart 1958, p. 157) the expansion reduces to a finite number of terms. This is true for any order of truncation of a Gram-Charlier series (see Cramer 1946, pp. 228-229). With an Edgeworth expansion of a Jacobi series, the series resulting from any order of truncation is not finite. The convergence of this series will be considered subsequent to the following algebraic development.

A3.5.2 The Jacobi Form of the Expansion

In order to simplify the Edgeworth expansion model, (2), (3) and (4) were developed for standardized r.vs. with $\mu_1 = 0$ and $\mu_2 = 1$. Thus in applying the model to the Jacobi series of A3.4(5) the expression A3.4(8) for the coefficients of the series must be converted into a standardized form. Observe first that

$$\mu_r\left(\frac{x-a}{b-a}\right) = \mu_r\left(\frac{\sigma(x)}{(b-a)}\left(\frac{x-\mu_1(x)}{\sigma(x)} - \frac{\mu_1(x)+a}{\sigma(x)}\right)\right)$$

$$= \left(\frac{\sigma(x)}{b-a}\right)^r \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} \left(\frac{\mu_1(x)+a}{\sigma(x)}\right)^{r-i} \mu_r\left(\frac{x-\mu_1(x)}{\sigma(x)}\right) \qquad (5)$$

Combining (5) with A3.4(8) leads to

$$D_n = \sum_{r=0}^{n} (-1)^r A_r^{(n)} \left(\left(\frac{\sigma(x)}{b-a}\right)^r \sum_{i=0}^{r} \binom{r}{i} \left(\frac{\mu_1(x)+a}{\sigma(x)}\right)^{r-i} \mu_r\left(\frac{x-\mu_1(x)}{\sigma(x)}\right)\right) \tag{6}$$

Through steps similar to those used to develop A3.4(19), equation (6) can be written as

$$D_n = \sum_{r=0}^{n} (-1)^r B_r^{(n)} \mu_r\left(\frac{x-\mu_1(x)}{\sigma(x)}\right) \tag{7}$$

where

$$B_r^{(n)} = \sum_{r=0}^{n} \binom{i}{r} \left(\frac{a+\mu_1(x)}{\sigma(x)}\right)^{i-r} \left(\frac{\sigma(x)}{b-a}\right)^r A_r^{(n)}$$

Since in (7), $\mu_1 = 0$ and $\mu_2 = 1$, the Edgeworth order n approximations of (2), (3) and (4) can be directly applied to (7). Note that for an order n approximation, $\mu_5\left(\frac{x-\mu_1(x)}{\sigma(x)}\right)$ is the lowest standardized moment to be approximated.

On carrying out these steps repetitively, one determines that $\hat{D}_n$, an Edgeworth order n approximation for D_n of (7) or A3.4(8), is given by

$$\hat{D}_n = \left[\sum_{\substack{r=0 \\ r \text{ even}}}^{n} \frac{r!\, B_r^{(n)}}{(\frac{r}{2})!\, 2^{r/2}}\right] - \left[\sum_{\substack{r=3 \\ r \text{ odd}}}^{n} \frac{r!\, B_r^{(n)}}{6\cdot(\frac{r-3}{2})!\, 2^{(r-3)/2}}\right]\mu_3$$

$$+ \left[\sum_{\substack{r=6 \\ r \text{ even}}}^{n} \frac{r!\, B_r^{(n)}}{72\cdot(\frac{r-6}{2})!\, 2^{(r-6)/2}}\right]\mu_3^2 + \left[\sum_{\substack{r=4 \\ r \text{ even}}}^{n} \frac{r!\, B_r^{(n)}}{24\cdot(\frac{r-4}{2})!\, 2^{(r-4)/2}}\right](\mu_4 - 3) \tag{8}$$

where $\mu_3 = \mu_3\left[\frac{x-\mu_1(x)}{\sigma(x)}\right]$ and $\mu_4 = \mu_4\left[\frac{x-\mu_1(x)}{\sigma(x)}\right]$

In particular, for n = 0, 1,..., 8 equation (8) for the order n expansion is

$$\hat{D}_0 = D_0 = B_0^{(0)} \qquad \hat{D}_1 = D_1 = B_0^{(1)} \qquad \hat{D}_2 = D_2 = B_0^{(2)} + B_2^{(2)} \tag{9}$$

$$\hat{D}_3 = D_3 = (B_0^{(3)}+B_2^{(3)}) - B_3^{(3)}\mu_3$$

$$\hat{D}_4 = D_4 = (B_0^{(4)}+B_2^{(4)}+3B_4^{(4)}) - B_3^{(4)}\mu_3 + B_4^{(4)}(\mu_4-3)$$

$$\hat{D}_5 = (B_0^{(5)}+B_2^{(5)}+3B_4^{(5)}) - (B_3^{(5)}+10B_5^{(5)})\mu_3 + B_4^{(5)}(\mu_4-3)$$

$$\hat{D}_6 = (B_0^{(6)}+B_2^{(6)}+3B_4^{(6)}+15B_6^{(6)}) - (B_3^{(6)}+10B_5^{(6)})\mu_3$$
$$+ 10B_6^{(6)}\mu_3^2 + (B_4^{(6)}+15B_6^{(6)})(\mu_4-3)$$

$$\hat{D}_7 = (B_0^{(7)}+B_2^{(7)}+3B_4^{(7)}+15B_6^{(7)}) - (B_3^{(7)}+10B_5^{(7)}+105B_7^{(7)})\mu_3$$
$$+ 10B_6^{(7)}\mu_3^2 + (B_4^{(7)}+15B_6^{(7)})(\mu_4-3)$$

$$\hat{D}_8 = (B_0^{(8)}+B_2^{(8)}+3B_4^{(8)}+15B_6^{(8)}+105B_8^{(8)}) - (B_3^{(8)}+10B_5^{(8)}+105B_7^{(8)})\mu_3$$
$$+ (10B_6^{(8)}+280B_8^{(8)})\mu_3^2 + (B_4^{(8)}+15B_6^{(8)}+210B_8^{(8)})(\mu_4-3)$$

The order $n^{1/2}$ Edgeworth expansion, based on the first three standardized moments, can be obtained from (8) and (9) by setting μ_3^2 and (μ_4-3) both equal to zero. Again the coefficients are not necessarily zero for large n, and questions of convergence are relevant. The discussion now turns to this issue.

A3.5.3 Convergence of the Edgeworth Expansion

In paragraph A3.4.2 it was seen that Jacobi series defined by A3.4(1) (or equivalently A3.4(7)) is numerically equivalent to a uniformly convergent series on the open interval (-1, 1). Thus, for any $y \varepsilon (0,1)$ or $x \varepsilon (a,b)$ the sequence $\{a_n\}$ of A3.4(7) converges. Now in order to manipulate freely the terms of $\{a_n\}$ and form other sequences that always converge to the same value, it is sufficient that the sequence $\{|a_n|\}$ of absolute values converges. Without absolute convergence, infinite subseries or rearrangements of $\{a_n\}$ and their subseries may converge to other values or not converge at all. In fact any conditionally convergent series of real terms can be rearranged to converge to any desired value (see Apostal 1957, pp. 367-369).

In order to generate from A3.4(7) an Edgeworth expansion with the infinite series specified by the approximation (8) two steps are necessary. First, parenthesis must be removed from the factor $D_n = (\sum_{r=0}^{n} (-1)^r \beta_r^{(n)} \mu_n)$ of A3.4(7) and second a subseries must be formed from the more detailed series formed in the first step. If in the first step the number of new terms introduced by removing each parenthesis is bounded independent of n and the resulting terms approach zero then removing parenthesis will preserve converges (see Apostal 1957, pp. 357-359). In general the second step can only be performed if the series resulting from the first step is absolutely convergent.

Equation (8) shows that the number of components of the Edgeworth expansion are not bounded as n increases. Consequently, the first step will not necessarily preserve convergence. With the numerous minus signs embedded in A3.4(7) there is no reason to believe that $\{a_n\}$ or a more detailed series formed by removing parenthesis will be absolutely convergent.

These comments are all directed at sufficient conditions for utilizing the convergence of $\{a_n\}$ in determining the convergence of related series. Since these sufficient conditions are not satisfied, the performance of each related series would have to be determined from more fundamental principles. This is not considered since the asymptotic characteristics of the expansion is of no particular consequence to the manner in which the Jacobi expansion could be used.

The usual Jacobi series is obviously preferable to the Edgeworth approximation when complete information is available about the moments of an unknown p.d.f. and the numerous terms of an asymptotic expansion are acceptable. Thus the Edgeworth expansion of a Jacobi series may be of value when information about moments is limited and/or a parsimonious representation is desired. Under either of these circumstances the asymptotic properties of the Edgeworth expansion are not particularly relevant. It is the initial and midrange properties of the expansion that are important when truncated series are used. Similar circumstances and conclusions have been discussed by Cramer (1946, pp. 223-224) for the Gram-Charlier series.

In these situations the relevant question is if the series should be truncated with, for example, the fourth moment, or if the unreliable higher moments should be approximated in accordance with a model of supplementary information. The Edgeworth expansion is, of course, only one model of such information. Regardless of the asymptotic properties of the expansion it is possible that the initial terms of the expansion may add meaningful information. Referring to (2) it is seen that the first approximation of an order n Edgeworth expansion occurs with the fifth moment. Thus, one must choose between using the unreliable fifth moment, not specifying its value, or using for example an order n Edgeworth approximation.

APPENDIX 4

PROPERTIES OF THE BETA-NORMAL DISTRIBUTION

A4.1 Preface

This appendix discusses the beta-normal total error distribution (Grimlund 1974; Felix and Grimlund 1977). The distribution's first four central moments, stated without proof in the latter paper, are derived in section A4.2. A new result giving noncentral moments of any order is also derived in section A4.2. Section A4.3 demonstrates that the beta-normal skewness and kurtosis approach the skewness and kurtosis of the component beta distribution when the total number of transactions, χ, is large. The kurtosis result is new. However, for completeness, both results are given. Section A4.3 uses these results to discuss the use of the gamma and extended beta distributions as approximations to the beta-normal distribution. It is assumed that the skewness and kurtosis are equal to their asymptotic (large χ) values.

The beta-normal distribution represents a continuous process that only assumes a nonzero value when one state of a Bernoulli process occurs. In auditing, this state can represent an error, a defective or obsolete item, a credit default, etc. Given that this state has occurred, there is then a second process representing the amount of the error, the required writedown, the amount of default, etc. It is assumed this latter process is normally distributed. The beta-normal distribution could be used to model customer behavior, natural resource exploration, or any other activity with two stages of uncertainty.

The Bayesian formulation of the procedure is particularly useful when low "error rates" are encountered in the preliminary Bernoulli process.* The procedure is based upon Bayesian natural conjugate analyses for the Bernoulli "error rate" process and the normal "error size" process. The resulting beta error rate distribution and the normal-gamma 2 distribution for the parameters the normal process are then consolidated into a single marginal distribution for the total error size in the population. This leads to the beta-normal density function defined by

$$f_{\beta N}(\pi_T) = \int_0^1 f_\beta(\rho|k,n) f_N(\pi_T|a\rho, 1/b\rho)d\rho \tag{1}$$

where $a\rho$ and $1/b\rho$ are the mean and precision of a marginal distribution for the total error amount in a population with error rate of ρ. Thus the variance of this distribution is $b\rho$.

The parameters a and b of (1) are defined by

$$a = \chi\mu_1(\pi) = \chi m \tag{2}$$

$$b = \chi\mu_2(\pi) = \chi\left(\frac{\nu}{\nu-2}\right)\left(\frac{1+k_n}{k_n}\right)v$$

where χ is the Bernoulli population size, and $\mu_1(\pi)$ and $\mu_2(\pi)$ are the expected value and variance of a student distribution for the marginal distribution of the size of individual errors. The parameters m, ν, v, and k_n are used to define the prior or posterior form of the normal-gamma 2 distribution.

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*The problem that motivates the beta-normal analysis is known in mathematical literature as a random sum of random variables. A non-Bayesian analysis for determining the characteristic function of such a random sum can be found in Feller (1966, p. 478). An expansion for the distribution function of a random sum has been developed by Marsh (1973).

A more detailed description of the beta-normal distribution can be found in the above references. A major inconvenience with the distribution is that the integral of (1) is not tractable. However, since the moments can be determined by switching the order of integration, a number of analytical approximations are possible. The results derived in the subsequent sections of this appendix are useful in this respect.

A4.2 Moments of the Beta-Normal Distribution

The moments of the beta-normal distribution can be calculated by switching the order of integration and utilizing the moment properties of the component normal and beta distributions.

A4.2.1 The First Four Central Moments

$$\mu_1(\beta N) = \int_{-\infty}^{\infty} \pi_T f_{\beta N}(\pi_T)\, d\pi_T = \int_0^1 f_\beta(\rho) \int_{-\infty}^{\infty} \pi_T f_N(\pi_T)\, d\pi_T\, d\rho$$

$$= \int_0^1 a\rho f_\beta(\rho)\, d\rho = a\mu_1(\beta) \quad \text{(or in condensed notation } a\mu_1\text{)} \tag{1}$$

The variance can be determined using the expansion $(\pi_T - a\mu_1)^2 = [(\pi_T - a\rho) + a(\rho-\mu_1)]^2$. Accordingly,

$$\mu_2(\beta N) = \int_{-\infty}^{\infty} (\pi_T - a\mu_1)^2 f_{\beta N}(\pi_T)\, d\pi_T$$

$$= \int_0^1 f_\beta(\rho) \int_{-\infty}^{\infty} [(\pi_T - a\rho)^2 + 2a(\pi_T - a\rho)(\rho-\mu_1) + a^2(\rho-\mu_1)^2]$$

$$\cdot\, f_N(\pi_T)\, d\pi_T\, d\rho$$

$$= \int_0^1 [b\rho + a^2(\rho-\mu_1)^2] f_\beta(\rho)\, d\rho = b\mu_1(\beta) + a^2\mu_2(\beta) \tag{2}$$

The third central moment follows from the expansion

References to several studies of the asymptotic properties of random sums also are given by Marsh.

$$(\pi_T - a\mu_1)^3 = (\pi_T - a\rho)^3 + 3a(\pi_T - a\rho)^2(\rho - \mu_1) + 3a^2(\pi_T - a\rho)(\rho - \mu_1)^2 + a^3(\rho - \mu_1)^3$$

Observing that the odd central moments of a normal distribution are zero and that

$$\rho(\rho - \mu_1) = [(\rho - \mu_1) + \mu_1](\rho - \mu_1) = (\rho - \mu_1)^2 + \mu_1(\rho - \mu_1)$$

it follows that

$$\mu_3(\beta N) = \int_0^1 f_\beta(\rho)[0 + 3ab\rho(\rho - \mu_1) + 0 + a^3(\rho - \mu_1)^3]d\rho$$

$$= 3ab\mu_2(\beta) + a^3\mu_3(\beta) \tag{3}$$

The fourth central moment follows from the expansion

$$(\pi_T - a\mu_1)^4 = (\pi_T - a\rho)^4 + 4a(\pi_T - a\rho)^3(\rho - \mu_1) + 6a^2(\pi_T - a\rho)^2(\rho - \mu_1)^2$$
$$+ 4a^3(\pi_T - a\rho)(\rho - \mu_1)^3 + a^4(\rho - \mu_1)^4$$

Noting that the kurtosis of a normal distribution is 3, and hence $\mu_4(N) = 3\mu_2^2(N)$, and that $\rho(\rho - \mu_1)^2 = (\rho - \mu_1)^3 + \mu_1(\rho - \mu_1)^2$ it follows that

$$\mu_4(\beta N) = \int_0^1 f_\beta(\rho)[3b^2\rho^2 + 0 + 6a^2b\rho(\rho - \mu_1)^2 + 0 + a^4(\rho - \mu_1)^4]d\rho$$

$$= 3b^2[\mu_2(\beta) + \mu_1^2(\beta)] + 6a^2b[\mu_3(\beta) + \mu_1(\beta)\mu_2(\beta)] + a^4\mu_4(\beta) \tag{4}$$

A4.2.2 The Noncentral Moments

A general expression for noncentral moments of the beta-normal distribution can be determined using the binomial expansion and the general expression for the central moments of a normal distribution (see Kendall and Stuart 1958, p. 60). From A4.1(1) the r^{th} noncentral moment is

$$\mu_r'(\beta N) = \int_{-\infty}^{\infty} \pi_T^r f_{\beta N}(\pi_T)d\pi_T = \int_0^1 f_\beta(\rho)\int_{-\infty}^{\infty} \pi_T^r \, f_N(\pi_T)d\pi_T d\rho$$

Now since

$$\pi_T^r = [(\pi_T - a\rho) + a\rho]^r = \sum_{k=0}^{r} \binom{r}{k} (\pi_T - a\rho)^k (a\rho)^{r-k} \tag{5}$$

$$\mu_k(N) = \frac{\sigma^k (k!)}{2^{k/2}[(k/2)!]} \qquad \text{for k even}$$

$$= 0 \qquad \text{for k odd}$$

it follows that

$$\begin{aligned} \mu_r'(\beta N) &= \sum_{\substack{k=0 \\ k \text{ even}}}^{r} \binom{r}{k} \frac{a^{r-k} b^{k/2}(k!)}{2^{k/2}[(k/2)!]} \int_0^1 \rho^{r-k} \rho^{k/2} f_\beta(\rho)\, d\rho \\ &= \sum_{\substack{k=0 \\ k \text{ even}}}^{r} \binom{r}{k} \frac{a^{r-k} b^{k/2}(k!)}{2^{k/2}[(k/2)!]} \mu_{r-k/2}'(\beta) \end{aligned} \tag{6}$$

where $\mu_{r-k/2}'(\beta)$ is defined by A2.2(1).

A4.3 Skewness, Kurtosis and Approximations to the Beta-Normal Distribution

In this section the asymptotic skewness and kurtosis of the beta-normal distribution for large χ are determined. These results are then used to study the robustness of approximations to the beta-normal distribution based on gamma and extended beta distributions.

A4.3.1 Skewness and Kurtosis

The following derivation shows that the skewness, $[\beta_1(\pi_T)]^{1/2}$, and the kurtosis, $\beta_2(\pi_T)$, of the beta-normal distribution approaches the skewness, $\sqrt{\beta_1}$, and kurtosis, β_2, of the component beta distribution as the number of transactions or accounts increases. The skewness of the beta-normal distribution is considered first.

It follows from A4.2(2) and A4.2(3) that

$$[\beta_1(\pi_T)]^{1/2} = \frac{\mu_3(\beta N)}{[\mu_2(\beta N)]^{3/2}} = \frac{a^3\mu_3+3ab\mu_2}{[b\mu_1+a^2\mu_2]^{3/2}} \tag{1}$$

where the moments μ_1, μ_2 and μ_3 are all with respect to the component beta distribution.

Observing that $\mu_3 = \sqrt{\beta_1}\mu_2^{3/2}$ equation (1) can be written as

$$[\beta_1(\pi_T)]^{1/2} = \left(\frac{a^2\mu_2}{b\mu_1+a^2\mu_2}\right)^{3/2}\sqrt{\beta_1} + \left(\frac{9a^2b^2\mu_2^2}{(b\mu_1+a^2\mu_2)^3}\right)^{1/2}$$

$$= \left(\frac{1}{1+(b/a^2)(\mu_1/\mu_2)}\right)^{3/2}\sqrt{\beta_1} + \left(\frac{9(b/a^2)\mu_2^2}{[(b/a^2)\mu_1+\mu_2]^3}\right)^{1/2} \tag{2}$$

From A4.1(2) it follows that $b/a^2 = [\mu_1(\pi)/\mu_2^2(\pi)]/\chi$. Hence for large χ

$$b/a^2 \cong 0 \tag{3}$$

Consequently from (2), $[\beta_1(\pi_T)]^{1/2} \to \sqrt{\beta_1}$ as $\chi \to \infty$.

From A4.2(2) and A4.2(4) it follows that the kurtosis of the beta-normal distribution is

$$\beta_2(\pi_T) = \frac{\mu_4(\beta N)}{\mu_2^2(\beta N)} = \frac{a^4\mu_4+6a^2b(\mu_3+\mu_1\mu_2)+3b^2(\mu_2+\mu_1^2)}{a^4\mu_2^2+2a^2b\mu_1\mu_2+b^2\mu_1^2}$$

$$= \frac{\mu_4+6(b/a^2)(\mu_3+\mu_1\mu_2)+3(b/a^2)^2(\mu_2+\mu_1^2)}{\mu_2^2+2(b/a^2)\mu_1\mu_2+(b/a^2)^2\mu_1^2} \tag{4}$$

Since from (3) $b/a^2 \cong 0$ for large χ, it follows from (4) that $\beta_2(\pi_T) \cong \mu_4/\mu_2^2$. This is just β_2, the kurtosis of the component beta distribution.

For a numerical example of these results discussed by Felix and Grimlund (1977), it can be shown that the beta-normal skewness converges faster than the beta-normal kurtosis to the limiting beta distribution values. For this example (with $p = 6$, $n = 120$ and $b/a^2 = 1/\chi$) the following comparisons can be made.

	$\sqrt{\beta_1(\pi_T)}$	$\beta_2(\pi)$
$\chi = 1000$	.749	2.63
$\chi = 5000$	.746	3.77
Limiting Value	.745	3.77

A4.3.2 Gamma and Beta Approximations to the Beta-Normal Distribution

In the examples considered by Felix and Grimlund (1977) gamma distributions were found to approximate quite adequately the beta-normal p.d.f. These observations are systematically investigated in this paragraph for large values of χ. The analysis leads to the conclusion that the robustness of the gamma approximation increases with higher values of n and decreases with lower expected values for the component beta distribution. Thus the gamma approximation may deteriorate somewhat with moderate beta priors over low error rates. It is suggested that the extended beta distribution be used as an approximation to the beta-normal distribution.

Since a gamma distribution is a limiting form of a beta distribution for which $2\beta_2 - 3\beta_1 = 2\mu_4/\mu_2^2 - 3\mu_3^2/\mu_2^3 = 6$ (see Elderton and Johnson 1969, pp. 40-46), an ordinal measure of the robustness of the gamma approximation for large values of χ is given by $(2\beta_2 - 3\beta_1)/6$. Thus, the approximation should perform best when this expression is near 1.

This expression can be converted into a more meaningful form by substituting the central moments from equations A2.2(4). This yields

$$(2\beta_2-3\beta_1)/6 = \frac{p(n-p)[2n^2+p(n-p)(n-6)]/n^4(n+1)(n+2)(n+3)}{p^2(n-p)^2/n^4(n+1)^2}$$

$$- \frac{2p^2(n-p)^2(n-2p)^2/n^6(n+1)^2(n+2)^2}{p^3(n-p)^3/n^6(n+1)^3}$$

$$= \left(\frac{n+1}{n+2}\right)\frac{1}{p(n-p)}\left[\frac{2n^2+p(n-p)(n-6)}{n+3} - \frac{2n^2-8p(n-p)}{n+2}\right] \tag{5}$$

If (n+3) is approximately equal to (n+2) then (5) reduces to $\left(\frac{n+1}{n+2}\right)$. Since this is in turn approximately equal to 1 a gamma distribution should adequately approximate the beta-normal distribution. The effect of the approximation can be further investigated by observing that

$$\left(\frac{1}{n+3}\right) = \left(\frac{1}{n+2}\right)\left(\frac{n+2}{n+3}\right) = \left(\frac{1}{n+2}\right)\left(1-\frac{1}{n+3}\right) = \frac{1}{n+2} - \frac{1}{(n+2)(n+3)} \tag{6}$$

Substituting (6) into (5) and observing that $\mu_1 = p/n$ it follows that

$$(2\beta_2-3\beta_1)/6 = \left(\frac{n+1}{n+2}\right)\left[1-\frac{2n^2+p(n-p)(n-6)}{p(n-p)(n+2)(n+3)}\right]$$

$$(2\beta_2-3\beta_1)/6 = \left(\frac{n+1}{n+2}\right)\left[1-\frac{2n^2+p(n-p)(n-6)}{p(n-p)(n+2)(n+3)}\right]\frac{-6)}{} \tag{7}$$

From (7) it is seen that small values of μ_1 can counteract the effect that a large n has on improving the accuracy of the gamma approximation to the beta-normal p.d.f. This effect was empirically observed by Felix and Grimlund.

It is suggested that the extended beta distribution defined by A2.1(1) be used to approximate the beta-normal p.d.f. The four parameters of this distribution allow the first four moments of the beta-normal to be fitted exactly. This is in contrast to the three moment fit of the gamma distribution. The asymptotic skewness and kurtosis properties discussed in paragraph A4.3.1 suggest that a truncated form of the beta-normal distribution is a scaled up standardized beta distribution for the χ values of greatest interest. This is, of course, exactly what an extended beta distribution is.

Paragraph A2.3.4 gives a simple algebraic procedure for calculating the parameters of the extended beta approximation given the first four central moments. Equations A4.2(1) through A4.2(4) can be used to determine the required central moments of the beta-normal p.d.f. These equations are based on the beta-normal constants a, b and central moments of the component beta distribution. These parameters can in turn be determined by using A4.1(2) and the standardized beta moment relation given by A2.2(4).

APPENDIX 5

A POISSON-GAMMA MODEL FOR THE COMPOSITION OF ERROR RATE AND SIZE UNCERTAINTY

A5.1 Preface

This appendix develops an alternative model to the beta-normal procedure of Felix and Grimlund (1977) for combining error rate and error size uncertainty. Rather than considering that errors are generated from Bernoulli and normal processes, Poisson and gamma processes are utilized. Individual errors are assumed to arise from a Poisson process with the size of the respective errors following a 2-parameter gamma distribution.

The primary advantage of this approach is that the gamma p.d.f. can represent both symmetric and nonsymmetric processes for the size of errors. However, the accompanying Poisson process with a single parameter does not provide as robust a representation of error intensity, as the beta p.d.f. of the beta-normal procedure.

The choice of the gamma distribution is somewhat arbitrary. Possible alternatives to the gamma distribution that also can represent skewness and mimic a normal distribution include the Weibull and the log-normal distributions (see Johnson and Kotz 1970a, pp. 117,253). These alternatives are reviewed in section A5.2.

Very little Bayesian theory has been developed for the 2-parameter gamma distribution process under the assumption that there is uncertainty about both the process skewness and scale (or variance). Lwin and Singh (1974) have developed a discrete prior to posterior analysis for the skewness parameter of the 2-parameter gamma distribution. These authors do not indicate any earlier work on this problem, a conclusion the present author also has reached.

In section A5.4 a natural conjugate joint density for both the skewness and scale parameters of the gamma distribution is developed. While the basic Bayesian properties of this new distribution are derived, the moment properties and possible forms of the distribution have yet to be investigated. These Bayesian properties of the distribution are used to model the auditor's prior and posterior uncertainty about the gamma error size distribution.

Section A5.3 discusses the use of the Poisson distribution as a model of error rate uncertainty. The implications of several different sets of assumptions are examined. In section A5.5 the error rate theory of section A5.3 is combined with the error size theory developed in section A5.4. This leads to several different total error distributions based upon Poisson and gamma distributions. These distributions correspond to the beta-normal distribution disc.ssed in appendix 4.

The results of this analysis are not as tractable as the equivalent beta-normal theory. There are also unresolved questions as to how these difficulties can be surmounted. Finally it is not immediately clear how these Poisson-gamma models can be integrated with a model of an i.c.s. The analyses of all these issues would be a major research task, perhaps worthy of future consideration. This appendix lays out the basic framework of this alternative approach. However, given the complexity of the audit process, no attempt is made in this dissertation to develop more completely this alternative model of the integration of audit evidence.

A5.2 Alternative Approaches

This section briefly considers the Weibull and lognormal distributions as two possible alternatives to the gamma distribution for recognizing skewness in the distribution of error sizes. There appears to have been very little work done on Bayesian prior to posterior procedures based upon either the Weibull or lognormal process. Thus the current discussion is limited to the analysis of Soland (1968,1969) and Kaufman (1963).

Soland developed a Bayesian analysis for a Weibull process. He showed that the lack of a sufficient statistic of fixed dimensionality rules out the possibility of constructing a natural conjugate distribution for the Weibull skewness parameter (see Raiffa and Schlaifer 1961, pp. 44-47). However, as shown by Soland (1969), it is possible to develop a discrete prior to posterior analysis for this parameter. These fixed dimensionally difficulties do not arise with gamma distributions. Thus, since the gamma distribution models approximately the same skewness and kurtosis configurations as Weibull distributions (see Rousu 1973), it seems preferable to the Weibull approach.

Another approach for recognizing a skewed distribution of error sizes is based upon the lognormal distribution with p.d.f.

$$f_L(x|\mu,h) = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{h}}{x}\exp\{-\tfrac{1}{2}h(\log x-\mu)^2\} \qquad (1)$$

where

$$0 < x < \infty \qquad 0 < h < \infty \qquad -\infty < \mu < \infty$$

The parameters μ and h are the mean and precision of the underlying normal p.d.f. of (1). The mean, μ_L, of the lognormal distribution is then given by

$$\log \mu_L = \mu + 1/2h \tag{2}$$

Assuming that the precision h is known, Kaufman (1963, pp. 161-162) showed that the lognormal distribution $f_L(\mu_L|m'+1/2h,n'h)$ is a natural conjugate prior for μ_L. This analysis can be extended to the more general case with the precision, h, unknown using the analysis of Raiffa and Schlaiffer (1961, pp. 300-301) for the normal distribution. It can be shown that a natural conjugate joint prior for (1) in the metric (μ,h) is given by Raiffa and Schlaiffer's normal-gamma 2 joint density.

In the metric (μ_L,h) the equivalent natural conjugate joint density is given by

$$f_{L\gamma 2}(\mu_L,h) = f_L(\mu_L|m'+1/2h,n'h)\; f_{\gamma 2}(h|v',\nu') \tag{3}$$

where $f_{\gamma 2}$ is a gamma 2 p.d.f. (Raiffa and Schlaiffer 1961, p. 226).

Several of the steps used to develop the beta-normal distribution are equally tractable with the lognormal distribution. However, a major difficulty arises in finding the distribution for the sum of r lognormal distributions. Or alternatively, the same type of problem arises in finding the distribution of the sum of r marginal error size distributions, where these marginal distributions have been found using the prior or posterior p.d.fs. to integrate out the lognormal parameters.

As will be seen, the gamma error size approach developed in section A5.4 avoids these summation difficulties. However, the natural conjugate relationship and several other steps of the gamma approach are not as tractable as these aspects of a lognormal approach. Thus the introduction of skewness with either the Weibull, lognormal or gamma distribution leads to analytically inconvenient results.

The focus of this appendix on the gamma distribution is motivated by such analytical considerations. Currently there is a complete lack of auditing research which might suggest what type of model of the error size process is most appropriate. While this appendix looks at skewness alternatives to normality, there is always the possibility that models that incorporate variations in kurtosis might be more appropriate.

A5.3 A Poisson Process and Sampling for Error Rates

A Poisson process can arise under a number of very different circumstances. This section examines these issues and develops an appropriate mathematical model for each set of assumptions.

A5.3.1 Physical Processes and Sampling Procedures

Two very different scenarios can be used to motivate a discussion of a Poisson process. First it can be assumed that there exists an ongoing physical process which generates errors such that one's uncertainty about the total number of errors to be generated corresponds to a Poisson mass function. Second, in certain circumstances it can be assumed that a sampling procedure for investigating an extant population's total number of errors generates a Poisson sampling process. In this second case it is not necessary to make any assumptions about the conditions under which the errors were generated.

In both cases the Poisson process usually arises as an approximation to a Bernoulli process with a very low error rate. In the first case it must be assumed that there is a constant probability of error affecting each item of the population. In contrast, the second case is not affected by a highly correlated error generating process. The population is just a pool of in error and not in error items. This leads to a constant probability that a given random sample item will be in error.

Corresponding to these two scenarios are two different objectives and resulting prediction models. In the first case interest focuses on predicting the error intensity of a production process. The second case is concerned with predicting the actual number of errors present in a collection of items generated by any process.

The external auditor's analysis of error rates usually corresponds to the second case. However, internal auditors, systems analysts and consultants may be interested in predicting future events rather than in just controlling their uncertainty about existing events. For these objectives the first case may be of interest. The validity of the predictions resulting from this case are, of course, dependent upon the error rate consistency of the process over time.

A5.3.2 An Ongoing Poisson Process

In the first case interest focuses on the unknown intensity, λ, of the process. The parameter λ can be thought of as the expected number of error items per unit. In an accounting environment it may be convenient to think of say $N_u = 1000$ elements per unit and avoid nonintuitive intensities of less than one. For a process generating χ units, or $N_u\chi$ elements, the probability mass function for the total number of items in error is given by

$$f_p(r|\lambda\chi) = e^{-\lambda\chi} (\lambda\chi)^r/r! \tag{1}$$

If each element of the process is generated by a Bernoulli process with a very small error rate, ρ , then (1) can represent a Poisson approximation to the binomially distributed total number of errors. Under such circumstances the expected value and variance of the binomial mass function are given by

$$E_b(r) = (N_u\chi)\rho \qquad Var_b(r) = (N_u\chi)\rho(1-\rho) \cong (N_u\rho)$$

Defining $\lambda = \rho N_u$, the Poisson mass function (1) has the same expected value and variance as the approximated values for the binomial mass function.

A5.3.3 Poisson Sampling of an Existing Population

A Poisson likelihood function can arise in this second case through a Poisson approximation to the binomial sampling process. Alternatively a gamma distribution approximation to a Pascal sampling process can lead to the Poisson likelihood function. In both cases the kernel of approximating likelihood function can be shown to be

$$\ell(\lambda|r,x) \propto e^{-\lambda x}\lambda^r \tag{2}$$

where r is the observed number of errors in x population units (or xN_u elements). In both cases the Poisson likelihood is only used to represent a sampling procedure. No assumption has been made about the nature of the process which originally generated the population.

The nature of the approximation is now considered for the Pascal case. For Pascal sampling, r is predetermined and a random number of observations, $n = xN_u$, is observed. The p.d.f. for this sampling process is

$$f_{Pa}(n|\rho,r) = \binom{n-1}{r-1}\rho^r(n-\rho)^{n-r} \tag{3}$$

where

$$E_{Pa}(n) = \frac{r}{\rho} \qquad Var_{Pa}(n) = \frac{r}{\rho^2}(1-\rho) \tag{4}$$

Now for $\rho << 1$, and $n = xN_u$ equations (4) become

$$E_{Pa}(x) = \frac{r}{\rho N_u} \qquad Var_{Pa}(x) \cong \frac{r}{(\rho N_u)^2} \tag{5}$$

Defining $\lambda = \rho N_u$ the moments of (5) are just those of a gamma p.d.f. for a continuous representation of the number, x, of units observed in order to locate r errors. The corresponding p.d.f. of x is

$$f_{\gamma_1}(x|r,\lambda) = \frac{\lambda^r}{\Gamma(r)} x^{r-1}e^{-\lambda x} \tag{6}$$

Thus rather than assuming a Pascal sample with the p.d.f. given by (3), it is assumed that there is a single observation x drawn from (6). The likelihood of this observation for fixed r is given by (2). Now the gamma p.d.f.

$$f_{\gamma_1}(\lambda|r',t') = \frac{\lambda^{r'-1}e^{-t'\lambda}}{(\frac{1}{t'})^{r'}\Gamma(r')}$$

is a natural conjugate prior to the likelihood kernal (2). Consequently, the posterior p.d.f. given this prior is

$$f_{\gamma_1}(\lambda|r'',t'') = \frac{\lambda^{r''-1}e^{-t''\lambda}}{(\frac{1}{t''})^{r''}\Gamma(r'')} \tag{7}$$

where $r'' = r' + r \quad t'' = t' + x$. This result is utilized in section A5.5 for combining error rate and error size information.

A5.3.4 Poisson Sampling of a Poisson Process

The prior two paragraphs have considered an ongoing Poisson process and sampling procedures that generate approximate Poisson processes. The composite case of Poisson sampling from a Poisson process is now briefly considered.

If λ is known then (1) represents one's uncertainty in the number of errors that a Poisson process will generate in processing χ units. More typically the past output units of a Poisson process are examined in order to estimate λ. Except for the likelihood now being an exact representation of the process, rather than an approximation, these estimating circumstances correspond to Poisson sampling. Equation (7) again gives a natural conjugate posterior distribution for λ.

Since the nature of the error generating process is assumed to be Poisson, the marginal distribution of r can be derived using (1) and (7). From Raiffa and Schlaifer (1961, p. 284) it follows that

$$f(r) = \int_0^\infty f_p(r|\lambda\chi) f_{\gamma 1}(\lambda|r'',t'')d\lambda$$

$$= f_{nb}\left(r\middle|\frac{\chi}{\chi+t''}, r''\right) = \binom{r+r''-1}{r''-1}\left(\frac{\chi}{\chi+t''}\right)^r\left(\frac{t''}{\chi+t''}\right)^{r''} \qquad (8)$$

Thus, the probability of r in error events in χ process units is given by a negative binominal distribution. This result will be utilized in section A5.5 for combining error number and error size information.

A5.4 A Gamma Distribution for Error Size

This section develops a prior to posterior analysis under the assumption that identified errors have error sizes distributed according to a gamma distribution. A natural conjugate distribution is developed for the skewness and scale parameters of the gamma process.

A5.4.1 The Basic Result

Assume that each error, π, is a realization from the gamma distribution

$$f_{\gamma_1}(\pi|\alpha,\beta,\gamma) = \frac{(\pi-\gamma)^{\alpha-1}}{\Gamma(\alpha)}\beta^{\alpha}e^{-(\pi-\gamma)\beta} \tag{1}$$

where γ is a known location parameter. Thus when r_s errors are observed, the likelihood function for the sample is

$$\ell(\alpha,\beta|\pi_1,\ldots\pi_{r_s}) = \frac{\pi_p^{\alpha-1}}{\Gamma(\alpha)^{r_s}}\beta^{\alpha r_s}\, e^{-\beta\pi_s} \tag{2}$$

where

$$\pi p = (\pi_1-\gamma)\ldots(\pi_{r_s}-\gamma) \text{ and } \pi_s = (\pi_1-\gamma)+\ldots+(\pi_{r_s}-\gamma) \tag{3}$$

A natural conjugate joint p.d.f. for the skewness, α, and scale, β, parameters of (1) is given by

$$f(\alpha,\beta|a',b',c',d') = f(\alpha|a',b',c',d')f_{\gamma_1}(\beta|\alpha c',d') \tag{4}$$

where

$$f(\alpha|a',b',c',d') = \frac{1}{I'}(k')^{\alpha}\frac{\Gamma(\alpha c')}{\Gamma(\alpha)^{b'}} \tag{5}$$

$$I' = \int_0^\infty (k')^\alpha \frac{\Gamma(\alpha c')}{\Gamma(\alpha)^{b'}}\, d\alpha \qquad k' = a'/(d')^{c'} \tag{6}$$

$$f_{\gamma_1}(\beta|\alpha c', d') = \frac{(d')^{\alpha c'}}{\Gamma(\alpha c')} \beta^{\alpha c'} e^{-\beta d'} \tag{7}$$

$$a' > 0 \quad b' > 0 \quad c' > 0 \quad d' > 0 \quad b' > c \quad 0 < k' < 1 \tag{8}$$

A5.4.2 The Natural Conjugate Relationship

Delaying for a moment questions of under what conditions the integral (6) exists, the natural conjugate relationships of (4) can be demonstrated by forming the product of (2) and (4)

$$f(\alpha,\beta)\ell(\alpha,\beta)$$

$$= \frac{a'}{I'} \frac{(a')^{\alpha-1}}{(d')^{\alpha c'}} \frac{\Gamma(\alpha c')}{\Gamma(\alpha)^{b'}} \cdot \frac{(d')^{\alpha c'}}{\Gamma(\alpha c')} \beta^{\alpha c'} e^{-\beta d'} \cdot \frac{\pi_p^{\alpha-1}}{\Gamma(\alpha)^{r_s}} \beta^{\alpha r_s} e^{-\beta \pi_s}$$

$$\propto \frac{(a'')^{\alpha-1}}{(d'')^{\alpha c''}} \frac{\Gamma(\alpha c'')}{\Gamma(\alpha)^{b''}} \quad \frac{(d'')^{\alpha c''}}{\Gamma(\alpha c'')} \beta^{\alpha c''} e^{-\beta d''} \tag{9}$$

where

$$a'' = a'\pi_p \qquad b'' = b' + r_s \tag{10}$$

$$d'' = d' + \pi_s \qquad c'' = c' + r_s$$

From (9) it can be seen that the kernel of the posterior joint density is of the same form as the prior joint density given by (5) and (7). Note that since $\pi_p > 0$, $\pi_s > 0$ and $r_s > 0$ equations (10) preserve the inequalities given by (8). In particular

$$k'' = \frac{a''}{(d'')^{c''}} = \frac{a'\pi_p}{(d'+\pi_s)^{c'}(d'+\pi_s)^{r_s}} < \frac{a'}{(d')^{c'}} \cdot \frac{\pi_p}{\pi_s^{r_s}} < \frac{\pi_p}{\pi_s^{r_s}} < 1$$

The last inequality follows from (3) on expanding $\pi_s^{r_s}$. Equations (10) indicate the sample equivalence of the prior constants a´, b´, c´, d´, and how an intuitive noninformative prior judgment can be established by setting a´ = 1 and b´ = c´ = d´ = 0 in the posterior density.

A5.4.3 The Existence and Calculation of the Skewness Integral

It is now shown that the improper integral (6) exists, and hence (5) is a proper p.d.f. It is only necessary to show that for large α^* the ratio of the gamma functions of (5) is less than one. Since $0 < k < 1$, it follows then from the convergence of

$$\int_0^\infty k^\alpha d\alpha = \int_0^\infty e^{-(-\log k)\alpha} d\alpha = 1/(-\log k)$$

that (6), with an integrand less than k for $\alpha > \alpha^*$, must also converge (see Buck 1956, p. 89).

That the required ratio is less than one can be shown using Stirling's Maclaurin series

$$\log\Gamma(x) = (x-\tfrac{1}{2})\log x - x + \tfrac{1}{2}\log 2\pi + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}B_k}{2k(2k-1)x^{2k-1}} \tag{11}$$

where B_1, B_2,... are Bernoulli numbers $\frac{1}{6}, \frac{1}{30}, \frac{1}{42} \ldots$

From (11) it follows that

$$\log\left[\frac{\Gamma(\alpha c)}{\Gamma(\alpha)^b}\right] = \log\Gamma(\alpha c) - b\log\Gamma(\alpha)$$

$$= \alpha\log\left(\frac{c^c}{\alpha^{b-c}}\right) - \frac{\log c}{2} + \frac{(b-1)\log\alpha}{2} + (b-c)\alpha$$

$$- (b-1)\log 2\pi + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}B_k}{2k(2k-1)\alpha^{2k-1}}\left(c^{-(2k-1)}-b\right) \tag{12}$$

Now for large α the first term of the infinite sum in (12) approaches 0. Since the terms of the summation are monotonically decreasing in absolute value, all subsequent terms also approach zero. Dropping these near zero terms (12) becomes

$$\cong \alpha[\log(\frac{c^c}{\alpha^{b-c}})+(b-c)] + \frac{(b-1)}{2}\log\alpha - \frac{\log c}{2} - (b-1)\log 2\pi \tag{13}$$

Since $b > c$, for large α the [...] term of (13) approaches $-\infty$. Now when $b < 1$ the second term of (13), $[(b-1)/2] \log \alpha$, also approaches $-\infty$. Consequently when $b < 1$, (13) and hence (12) approach $-\infty$. When $b \geq 1$, it follows for large α that $[(b-1)/2] \log \alpha < \alpha$ and hence (13) is

$$\leq \alpha[\log(\frac{c^c}{\alpha^{b-c}}) + (b-c) + 1] \tag{14}$$

Since for large α (14) approaches $-\infty$, (12) also approaches $-\infty$. Thus for $b > c$ the gamma function ratio approaches 0 and the integral (6) converges.

Equation (12) can also be used to express the normalization constant I of (6) in a more compact form for numerical integration. Writing (6) in exponential form (without prime notation) and substituting (12) yields

$$\begin{aligned} I &= \int_0^\infty \exp\{\alpha\log + \log\left[\frac{\Gamma(\alpha c)}{\Gamma(\alpha)^b}\right]\}d\alpha \\ &= \frac{1}{k_1}\int_0^\infty \exp\{k_2\alpha + [-(b-c)\alpha + (b-1)/2] \log\alpha + S(\alpha)\}d\alpha \end{aligned} \tag{15}$$

where

$$k_1 = (2\pi)^{b-1}c^{1/2}$$

$$k_2 = \log k + \log c^c + (b-c) = (b-c) + \log a \left(\frac{c}{d}\right)^c$$

$$S(\alpha) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1} B_i}{2i(2i-1)} \left(\left(\frac{1}{c}\right)^{2i-1} - b\right)\left(\frac{1}{\alpha}\right)^{2i-1}$$

The exponential form of (15) will eliminate some of the numerical problems associated with evaluating gamma functions. The author has found this same technique to be useful in evaluating beta and gamma p.d.fs.

A5.5 Total Error Distributions

The analysis of this section combines the results of section A5.3 for the number of errors and error rates with the error size results of section A5.4. The analysis proceeds in three steps: the determination of the conditional error size distribution given r errors and parameters α and β; the determination of the marginal distribution for r errors; and finally the determination of the unconditional total error distributions.

A5.5.1 The Conditional Distribution

The conditional error size distribution for r errors with gamma parameters α and β is easily shown to be

$$f_{\gamma_1}(\pi_T | r\alpha, \beta, r\gamma) = \frac{(\pi_T - r\gamma)^{r\alpha-1} e^{-(\pi_T - r\gamma)\beta}}{\Gamma(r\alpha) \left(\frac{1}{\beta}\right)^{r\alpha}} \tag{1}$$

where

$$\pi_T = \pi_1 + \ldots + \pi_r = (\pi_1 - \gamma) + \ldots + (\pi_r - \gamma) + r\gamma$$

is the sum of r identically distributed r.vs. with p.d.f. defined by A5.4(1). This result is derived using the convolution property of gamma distributions with identical scale parameters and location parameters of zero (Raiffa and Schlaifer 1961, p. 225). A linear transformation is utilized before and after the application of the convolution property.

Equation (1) is conditional dependent on α, β and r. Using the prior or posterior forms of the p.d.f. for α,β given by A5.4(4) through A5.4(10) it follows that

$$f(\pi_T|r) = \int_0^\infty\int_0^\infty f_{\gamma_1}(\pi_T|r\alpha,\beta,r\gamma)f_{\gamma_1}(\beta|c'\alpha,d') \cdot f(\alpha|a',b',c',d')\,d\beta\,d\alpha \tag{2}$$

While the parameters are destinated with prime notation the results are, of course, applicable to both prior and posterior p.d.fs.

It is easily shown (see Raiffa and Schlaifer 1961, pp. 221, 279) that (2) reduces to

$$f(\pi_T|r)\int_0^\infty f_{i\beta_2}(\pi_T-r\gamma|r\alpha,c'\alpha,d')f(\alpha|a',b',c',d')\,d\alpha \tag{3}$$

where

$$f_{i\beta_2}(\pi_T-r\gamma) = \frac{(d')^{c'\alpha}}{B(r\alpha,c'\alpha)}\,\frac{(\pi_T-r\gamma)^{r\alpha-1}}{(\pi_T-r\gamma+d')^{(r+c')\alpha}} \tag{4}$$

Substituting the p.d.f. for $f(\alpha)$ given by A5.4(5) equation (3) becomes

$$f(\pi_T|r) = \int_0^\infty \frac{(a')^{\alpha}\Gamma((r+c')\alpha)}{I\ \Gamma(r\alpha)\Gamma(\alpha)^{b'}}\,\frac{(\pi_T-r\gamma)^{r\alpha-1}}{(\pi_T-r\gamma+d')^{(r+c')\alpha}}\,d\alpha \tag{5}$$

Using Stirling's expansion A5.4(11), equation (5) can be expressed as

$$f(\pi_T|r) = \left[\frac{(\pi_T-r\gamma)^{-1}}{I}\left(\frac{r}{r+c'}\right)^{1/2}\right]\int_0^\infty\left[\frac{a'(\pi_T-r\gamma)^r}{(\pi_T-r\gamma+d')^{r+c'}}\right]^\alpha$$

$$\cdot\ \exp\{[(b'-c')+(r+c')\log(r+c')-r\log r]\ \alpha + [c'-b']\alpha\log\alpha + [\tfrac{b'}{2}]\log\alpha$$

$$+ \sum_{k=1}^{\infty}\left[\frac{(-1)^k B_k}{2k(2k-1)}\left(\left(\frac{1}{a'}\right)^{2k-1} - b' - \left(\frac{1}{r}\right)^{2k-1}\right)\right]\left(\frac{1}{\alpha}\right)^{2k-1}\}\ d\alpha \qquad (6)$$

Note that in a numerical evaluation with fixed r and π_T all the [...] terms are constant.

Equation (5) summarizes the marginal uncertainty in the aggregate size of r errors given a gamma error generating process and a natural conjugate prior/posterior analysis. Note that when α is known or discretely estimated (5) can be replaced by the more tractable inverted beta 2 distribution given by (4).

A5.5.2 The Unconditional or Marginal Distribution

Given a probability mass function for r, the unconditional (on r) total error distribution can be determined from

$$f(\pi_T) = \sum_{r=0}^{\infty} f(r)f(\pi_T|r) \qquad (7)$$

where $f(\pi_T|r)$ is defined by (5) or the series expansion (6).

Equation (7) can be applied whenever the uncertainty in r is the result of a Poisson process with known intensity λ. For this case it follows from A5.3(1) that*

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*Moments for this case can be found by differentiating the characteristic function of the compound Poisson distribution (Ross 1970, p. 23). Accordingly, $\phi\pi_T(t) = \exp[\lambda\chi(\phi_\pi(t)-1)]$ where $\phi_\pi(t)$ is the well known characteristic function of A5.4(1).

$$f(r) = f_p(r|\lambda\chi) = e^{-\lambda\chi}(\lambda\chi)^r/r! \tag{8}$$

Similarly, when a Poisson sample is taken from a Poisson process, as discussed in paragraph A5.3.4, (7) can be directly applied using A5.3(8).

The discrete, integer domain, of r in (7) is incompatible with the continuous p.d.f. $f(\lambda)$ for error intensity developed in paragraph A5.3.3. This p.d.f. arises as a result of a Poisson approximation to the likelihood function when sampling an existing population. The required p.d.f. $f(r)$ of (7) could be determined for this case by splitting up $f(\lambda)$ into a mass function for each integer value of r. However, since $f(\lambda)$ is a continuous approximation to a discrete process, it also seems appropriate to approximate the discrete values of r in $f(\pi_T|r)$ by a continuous function. Thus, for a population of N_p elements or N_p/N_u units it is assumed that

$$r = \lambda\frac{N_p}{N_u} \tag{9}$$

Using (5), (9) and A5.3(7) it follows that the total error distribution for sampling an existing population is

$$\begin{aligned} f(\pi_T) &= \int_0^\infty f(\pi_T|\lambda\frac{N_p}{N})f_{\gamma_1}(\lambda|r',t')d\lambda \\ &= \int_0^\infty f(\pi_T|r)f_{\gamma_1}(r|r',\frac{N_u}{N_p}t')dr \end{aligned} \tag{10}$$

where r, the dummy variable of integration in (10) resulting from the change of variable $r = \lambda\frac{N_p}{N_u}$, is no longer confined to integer values.

The symbolic notation of (10) masks the embedded double integration over both α and r. Thus, the calculation of a cumulative probability requires triple integration. While not a particularly attractive possibility, numerical computer procedures are available for evaluating such multiple integrals. The dimensionality of this integration is reduced by one when α is known or discretely estimated, and consequently $f(\pi_T|r)$ is the inverted beta 2 p.d.f. defined by (4).

Equation (10) corresponds to the beta-normal p.d.f. of Felix and Grimlund (1977). However, unlike the beta-normal p.d.f., analytical expressions for the moments of (10) cannot be found by switching the order of integration. Thus, it is not possible to derive an approximation based upon moments as was developed for the beta-normal p.d.f. While these observations may not be particularly significant when a single solution is desired, they may seriously limit the cost effectiveness of the Poisson-gamma model when extensive sensitivity is required. Additional development of the model, or technological innovations in computation may, of course, temper these observations.

VITA

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